Schur-Weyl duality \( V \cong \text{f.d. vector space} \)

\( V \otimes n \) is rep of \( GL(V) \)
also rep of \( S_n \) symmetric group

\[
\sigma(V_1 \otimes \cdots \otimes V_n) = V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}
\]

Representation structures give \( \left( U = \text{End}(V \otimes n) \right) \)

\( GL(V) \to U \), \( A = \text{linear span of image} \)
\( C \)-subalgebra of \( U \)

\( S_n \to U \), \( B = \text{linear span of image} \)
\( C \)-subalgebra of \( U \)

**Def:** Given subalg. \( R \subset \text{End}(W) \), let
\[
R^1 = \{ x \in \text{End}(W) \mid x r = r x \ \forall r \in R \}
\]

\( A \subseteq B^1 \), \( B \subseteq A^1 \).
Prop \[ B' = A \]
\[ \text{Pf.} \quad U = \text{End}(V \otimes n) = (\text{End}(V))^\otimes n \] as follows:

Given \( A_1 \otimes \cdots \otimes A_n, \ A_i \in \text{End}(V), \) get

\[ \bigotimes^n V \longrightarrow \bigotimes^n V \text{ by } V_1 \otimes \cdots \otimes V_n \mapsto A_1(V_1) \otimes \cdots \otimes A_n(V_n) \]

Under this identification, image of \( g \in \text{GL}(V) \) in \( U \) goes to \( g \circ g \circ \cdots \circ g \in (\text{End} V)^\otimes n \)

\[ A = \{ \varphi \in (\text{End} V)^\otimes n \mid \sigma \varphi = \varphi \ \forall \sigma \in G_n \} \]

Need to identify Symmetric tensors in \( \text{End}(V \otimes n) \)

Pick \( A_1 \otimes \cdots \otimes A_n \in \text{End}(V)^\otimes n, \ \sigma \in G_n \)

\( \sigma(A_1 \otimes \cdots \otimes A_n) = A_{\sigma^{-1}(1)} \otimes \cdots \otimes A_{\sigma^{-1}(n)} \)

in \( \text{End}(V \otimes n) \), this corresponds to

\[ V_1 \otimes \cdots \otimes V_n \mapsto A_{\sigma^{-1}(1)}(V_1) \otimes \cdots \otimes A_{\sigma^{-1}(n)}(V_n) \in V \otimes n \]

\[ = \sigma(A_i(V_{\sigma(i)}) \otimes \cdots \otimes A_n(V_{\sigma(n)})) \]

\[ = \sigma \circ (A_1 \otimes \cdots \otimes A_n) \circ \sigma^{-1}(V_1 \otimes \cdots \otimes V_n) \]

\( \exists \alpha \) being symmetric in \( \text{End}(V)^\otimes n \) translates to

\[ \alpha = \alpha \ \forall \sigma \in G_n \text{ in End}(V \otimes n) \Rightarrow \alpha \in B' \]
Being symmetric in $\text{End}(V)^\otimes n$ is equivalent to being in $B$ or $\text{End}(V^\otimes n)$. \(\Rightarrow A = B \triangleleft \square \)

Cor. $A^! = B$

pf. $B = \text{quotient of group algebra of } S_n$

$C(C(S_n))$ is semisimple $\Rightarrow B$ semisimple

use double commutant thm \(\square\)

Cor. As $S_n \times GL(V)$-representation, get

$V^\otimes n \cong \bigoplus S^\lambda \otimes S_\lambda(V)$

where $S^\lambda = \text{irred. } S_n$-rep.

Pf. By Pieri's rule, $V^\otimes n$ decomposes as sum of $S^\lambda(V)$ where $\ell(\lambda) \leq \dim V$ and $|\lambda| = n$ w/multiplicity given by sequences

$\lambda \subseteq \lambda^{(2)} \subseteq \lambda^{(3)} \subseteq \ldots \subseteq \lambda^{(n)} = \lambda$

where $\lambda^{(i+1)}/\lambda^{(i)} = \text{one box } \lambda^{(i)}$ partitions.

For every $\lambda$, at least one such sequence exists. $\square$
Rule \( S^1 \leftarrow \text{Specht modules} \)

\[ \dim = \text{mult of } S_{\lambda}(V) = \# \text{tableau of shape } \lambda \text{ using } 1, \ldots, n, \text{ each exactly once} \]

(i.e., standard Young tableaux)

Next goal: replace \( GL(V) \) by either \( O(V), \text{Sp}(V) \) that makes \( A \) smaller, and necessarily \( B \) gets larger (quotients of Brauer algebras)

First fundamental theorem of invariant theory (FFT)

setup: \( E, V \) vector spaces, \( \dim V = m, \dim E = n \)

\( G \subset GL(V) \). Consider ring of invariants

\[ \text{Sym}(V \otimes E)^G = \{ f \in \text{Sym}(V \otimes E) \mid gf = f \; \forall g \in G \} \]

\[ \bigoplus_{\lambda} S_{\lambda}(V) \otimes S_{\lambda}(E) \]

\( GL(E) \) commutes with \( GL(V) \) (and hence \( G \)), so \( GL(E) \) acts on \( \text{Sym}(V \otimes E)^G \).
Lemma. If \( \dim V = m \), & \( n \geq m \), then
\[
\text{Sym}(V \otimes \mathbb{C}^n)^G \text{ is generated by } \text{Sym}(V \otimes \mathbb{C}^m)^G \text{ together w/ action of } \text{GL}_n \mathbb{C}.
\]

If \( \text{Sym}(V \otimes \mathbb{C}^n)^G = \bigoplus S_{\lambda}(V)^G \otimes S_{\lambda}(\mathbb{C}^n) \) \( \leq \min(n,m) = m \)

So h.m. vector for \( S_{\lambda} \mathbb{C}^n \) only uses first \( m \) basis vectors. \( \Rightarrow \) \( S_{\lambda} \mathbb{C}^n \) is generated by \( S_{\lambda} \mathbb{C}^m \) together w/ action of \( \text{GL}_n \mathbb{C} \).

\( \Rightarrow \) \( S_{\lambda}(V)^G \otimes S_{\lambda}(\mathbb{C}^n) \) is generated by \( S_{\lambda}(V)^G \otimes S_{\lambda}(\mathbb{C}^m) \) together w/ \( \text{GL}_n \mathbb{C} \). \( \square \)

Consider condition \( G \subseteq \text{SL}(V) \), i.e., \( \det g = 1 \) \( \forall g \in G \).

If we pick bases of \( V, E \), can identify \( V \otimes E \) as \( n \times m \) matrices. The determinants of \( m \times m \) submatrices are invariant under \( SL(V) \), hence \( G \).

Lemma. If \( G \subseteq \text{SL}(V) \), and \( n \geq m \), then \( \text{Sym}(V \otimes \mathbb{C}^m)^G \) is gen. by \( \text{Sym}(V \otimes \mathbb{C}^{m-1})^G \) together w/ \( m \times m \) determinants and action of \( \text{GL}_n \mathbb{C} \).
**Pf.** When \( t=(1^n) \), \( S_2^\chi V = \wedge^m V \) is the determinant rep. Furthermore, if \( \lambda(\mu) = m \), then \( \mu = (k^m) + \nu \) where \( \lambda(\nu) < m \). (\( k = \mu_m \))

\[
\begin{bmatrix}
  k^m \\
  V
\end{bmatrix} \Rightarrow S_m C^m \cong \text{det}(\nu)^{\otimes k} \otimes S_k C^m
\]

h.w. vector for \( S_m C^m \) is h.w. vector for \( S_k C^m \) times \( \text{det}^k \)

\[
\Rightarrow (S_m V)^G \otimes S_m C^m \text{ is generated by}
\]

\[
(S_2 V)^G \otimes S_2 C^m \text{ by multiplying by } \text{det}^k. \]

\[\square\]

Now consider \( G = O(V) \) w/ form \( \psi \).

\( \psi \) gives identification \( V \cong V^* \): given \( v \in V \),

get linear functional \( u \mapsto \psi(u,v) \). This gives \( G \)-invariant subspace of \( \text{Sym}^2 V \) spanned by \( \psi \).

\[\Rightarrow \text{ get } G\text{-inv. subspace in } \text{Sym}^2 V \otimes \text{Sym}^2 E \text{ spanned by } \psi \otimes e, e', e'' \in E.\]
More explicitly, this is spanned by $\phi_{ij}$ $(1 \leq i \leq j \leq n)$ defined by

$$
\phi_{ij} \left( \sum_{k=1}^{n} v_k \otimes e_k \right) = \omega(e_i, e_j)
$$

As matrices, $\phi_{ij}$ applies $\omega$ to the $i$th and $j$th columns. If $j < i$, define $\phi_{ij} = \phi_{ji}$.

**Thm (FFT for $SO_0$)** Let $dim \; V = m$.

1. $\text{Sym}(V \otimes \mathbb{C}^n)^{SO_0(V)}$ is gen. as $C$-algebra by $\phi_{ij}$ and $m \times m$ determinants (if $m \leq n$)

2. $\text{Sym}(V \otimes \mathbb{C}^n)^{O(V)}$ is gen. as $C$-algebra by $\phi_{ij}$.

**PF** Let $R_n$ be $C$-algebra gen. by $\phi_{ij}$ and $m \times m$ determinants. Note: $R_n$ is closed under action of $\text{GL}_n \mathbb{C}$ (since $\omega$ is bilinear).
Step 1. Show that $\odot \Rightarrow \boxdot$. Sufficient to assume $n = m$ ($G_{n} \subset R_{m} = R_{n}$) $g \in O(V)$ acts on $m \times m$ determinants by $\pm 1$
So, assuming $\odot$ holds, $Sym(V \otimes C^{m}) O(V)$ is gen. by $\varphi_{ij}$ and products of even # of $m \times m$ determinants
If $u$ is the generic $m \times n$ matrix, we have $u^{	op} I' u = (\varphi_{ij}) = \Phi$
$\Rightarrow \det \Phi = \det (u)^{2} \cdot (-1)^{m}$
$\det \Phi$ gen by $\varphi_{ij} \Rightarrow \det (u)^{2}$ gen by $\varphi_{ij}$
$\Rightarrow (\det u)^{2k}$ gen by $\varphi_{ij}$

Hence $\odot \Rightarrow \boxdot$. 