Then (FFT for $\mathfrak{so}(n)$) Let $\dim V = m$.

1. $\text{Sym}(V \otimes \mathbb{C}^n) \mathfrak{so}(V)$ is gen. as $C$-algebra by $q_{ij}$ and $m \times m$ determinants (if $m \leq n$)

2. $\text{Sym}(V \otimes \mathbb{C})^{O(V)}$ is gen. as $C$-algebra by $q_{ij}$.

Pf. $R = \text{ring gen by } q_{ij}$ & $m \times m$ determinants

Last time showed $\text{(1)} \Rightarrow \text{(2)}$

Also can assume $n = m - 1$ to prove $\text{(1)}$

Now, prove $\text{(1)}$ by induction on $m$.

Base case $m = 1$: nothing to show.

Now assume $m > 1$. Pick $v \in V$ s.t. $w(v, v) = 1$.

$V' = \langle v \rangle^\perp$, so $V = \langle v \rangle \oplus V'$
If $g \in O(V')$, can extend to element $\tilde{g} \in SO(V)$ by $\tilde{g}(u) = det g \cdot u$.

If $f \in Sym(V \otimes \mathbb{C}^n)^{SO(V)}$, then restriction $f' \in Sym(V' \otimes \mathbb{C}^n)^{SO(V')}$ by induction, $f' = polynomial in q_{ij}$.

Let $F$ be same polynomial in $q_{ij}$ inside $Sym(V \otimes \mathbb{C}^n)^{SO(V)}$.

If $u'$ is any other vector of norm 1, then $\exists g \in SO(V)$ s.t. $g(u) = u'$ & hence $F - f$ restricted to $Sym(\langle u' \rangle \otimes \mathbb{C}^n)$ is 0. The set of vectors of nonzero norm is Zariski dense in $V$ $\Rightarrow$ $F - f$ is 0 on this set $\Rightarrow$ $F = f$. $\square$
$V^\otimes n \cong O(V)$. 

If $n = 2k$ even, pick a perfect matching $M$ of $\{1, \ldots, n\}$, i.e., write $\{1, \ldots, n\}$ as disjoint union of 2-element subsets $\sim$.

$\varphi : V^\otimes n \to C$ linear, $O(V)$-invariant.

$\varphi_m (v_1 \otimes \cdots \otimes v_n) = \prod_{\{i:j\} \in m} \omega (v_i, v_j).$

$\varphi_m \in (V^\otimes n)^* \cong V^\otimes n$ since $\omega$ gives identification $V \cong V^*.$

Cor.

$(V^\otimes n)^* \circ (V) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \text{spanned by } \varphi_m & \text{if } n \text{ even} \end{cases}$
\[ \text{Pf. } \text{Sym}(V \otimes \mathbb{C}^n) = \text{Sym}(V \otimes \mathbb{C}) - \text{Sym}(V) \otimes \mathbb{C}^n \]

Contains \( V \otimes \mathbb{C}^n \) as a linear subspace, namely the \((1, \ldots, 1)\)-weight space under the maximal torus in \( \text{GL}_n(\mathbb{C}) \).

\[ \text{FFT} \Rightarrow \mathcal{O}(V)\text{-invariants of } \text{Sym}(V \otimes \mathbb{C}^n) \]

gen. by \( \psi_{ij} \). \(-\) weight vector of weight 
\[
(0, \ldots, 1, \ldots, 0)
\]

\[
\Rightarrow (V \otimes \mathbb{C}^n)^{\mathcal{O}(V)} \text{ spanned by products of } \psi_{ij} \text{ w/ total weight } (1, \ldots, 1) .
\]

i.e., \( \mathcal{O}_M \), overall perfect matching \( M \) of \( \{1, \ldots, n\} \).
Now consider $G = Sp(V)$, $V$ symplectic $\omega$ from $\omega$. For $1 \leq i < j \leq n$, have $\Psi_{ij} \in Sym( V \otimes \mathbb{C}^n)$ given by

$$\Psi_{ij}( \sum v_i \otimes e_i ) = \omega( v_i, v_j ).$$

If $i \leq j$, define $\Psi_{ij} = -\Psi_{ji}$, $\Psi_{ii} = 0$.

Thm (FFT for $Sp(V)$): $Sym( V \otimes \mathbb{C}^n )_{Sp(V)}$ is generated as $C$-algebra by $\Psi_{ij}$.

Pf. $R = C$-algebra gen. by $\Psi_{ij}$.

$R$ closed under $GL_n \mathbb{C} \Rightarrow$ software to consider $n = \dim V$, assume that.

Let $u$ be generic matrix. Then

$$u^T \Omega u = (\Psi_{ij}) = \overline{\Omega} \in$ skew-

symmetric

$\text{Pf}(\overline{\Omega}) = \text{Pf}(\overline{\Omega}) \det(u) = \pm \det(u)$
pf(I) is polynomial in $y_{ij} \Rightarrow pf(y) \in R$

$\Rightarrow det(u) \in R$, so $R$ already contains

$m \times m$ determinants.

$\Rightarrow$ can assume $m = \dim u - 1$.

Idea: do induction on $\dim u$.

Induction step similar to orthogonal case.

Details in notes. \qed

$V \otimes^n \mathfrak{sp}(u)$. If $n = 2k$ even, pick perfect

matching $M$ of $\{1, \ldots, n\}$, define

$y_M : V \otimes^n \rightarrow \mathfrak{c}$

$y_M (v_1 \otimes \ldots \otimes v_n) = \prod_{\{i < j\} \in M} w(v_i, v_j)$

As before:

$Cor. (V \otimes^n)^{sp}(u) = \begin{cases} 
0 & \text{if odd} \\
\sum_{\text{spanned by } y_M} & \text{if even.}
\end{cases}$
Brauer algebra.

Goal: compute commutant of $G \leq \text{End}(V_{\otimes n})$ where $G = \text{O}(V)$ or $G = \text{Sp}(V)$.

$\text{End}(V_{\otimes n}) = V_{\otimes n}^* \otimes (V_{\otimes n})^* \cong V_{\otimes 2n}$

commutant of $G$ is identified w/ $(V_{\otimes 2n})^*$

Given a perfect matching $M$ of $\{1, \ldots, 2n\}$, get $\chi_M / \chi_M \in \text{End}(V_{\otimes 2n})$ and $\beta_M \in \text{End}(V_{\otimes n})$.

acts on $V_{\otimes n}$ as follows:

1. Given edge $\{i, j\}$ w/ $i, j \leq n$, we remove $V_i, V_j$ and scale result $w(V_i, V_j)$.

2. Given edge $\{i, j > n\}$ w/ $i, j \leq n$, then move $V_i$ to the $j$th tensor position.

(The set of edges forms a bijection between a subset of $\{1, \ldots, n\}$ and subset of $\{n+1, \ldots, 2n\}$.

In symplectic case, multiply by $\text{sign}$ of this bijection.
(3) For edges \( \mathfrak{o}_{i+n,j+n} \), \( i,j \leq n \), we insert element \( \mathfrak{s} \) in position \( i,j \) where 
\[ \mathfrak{s} \in \mathbb{O}_2 \] is given by
\[
\mathfrak{s} = \sum_{i=1}^{n} e_i \otimes e_{m+1-i} \quad \{ e_1, \ldots, e_m \} \text{ hyperbolic basis in } O(n) \text{ case}
\]
\[
\mathfrak{s} = \sum_{i=1}^{n} e_i \otimes e_{m+1-i} + e_{m+1-i} \otimes e_i \\
\{ e_1, \ldots, e_m \} \text{ symplectic basis for } Sp(n) \text{ case.}
\]

**Ex.** \( G = O(n) \), \( n = 4 \).

\[ \\
\begin{array}{c}
\begin{array}{cccc}
5 & 6 & 7 & 8 \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{array}
\]

\[
\beta_M(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = \omega(v_2, v_3) \sum_{i=1}^{m} v_i \otimes e_i \otimes v_4 \otimes e_{m+1-i}
\]
The commutant \( B = A^1 \) is linearly spanned by \( \beta_m \), \( m \) ranges over all perfect matchings of \( \{1, \ldots, 2n\} \).

We first define product structure on the set \( B_{n+1} \) of perfect matchings of \( \{1, \ldots, 2n\} \).

\[
\begin{array}{c}
M \\
\rightarrow \\
\begin{array}{c}
1 \\
2 \\
\vdots \\
n
\end{array}
\end{array}
\]

To multiply \( M, M' \):

1. Stack \( M' \) on top of \( M \)
2. Identify the middle row of vertices
3. Get reduced perfect matching on union of bottom row & top row. Result is \( MM' \).

\[ n(M, M') = \text{number of closed loops in middle row (which is discarded)} \]
Ex. \( n = 6 \)

\[ M: \]

\[ M': \]

stack then:

\[ MM': \]

\[ n(m, m') = 1 \]
Def. Pick $S \subseteq C$. The Bauer algebra $B_n(S)$ has basis given by perfect matchings $e_M \text{ w/ product}$

$$e_M \cdot e_{M'} = \sum_{n(M,M')} MM'$$

(0° = 1)

Then. We have surjective algebra homomorphisms

$$B_n(\dim V) \rightarrow \text{End}_{O(n)}(V \otimes^n)$$

$$B_n(-\dim V) \rightarrow \text{End}_{Sp(n)}(V \otimes^n)$$

which sends $e_M \mapsto \beta_M$.

(If $\dim V \gg n$, then, these are isomorphisms)

by double commutant, image is semisimple, but $B_n(S)$ need not be semisimple!