Setup: $V$ orthogonal space of dim $m$, 
$\omega$ orthogonal form.

$$T(V) = \bigoplus_{d \geq 0} V \otimes_d \omega$$

with concatenation of tensors as product

(tensor algebra)

$C(V, \omega) = C(V)$ is the quotient of $T(V)$
by 2-sided ideal generated by relations

$$v w + w v = 2 \omega(v, w) \quad v, w \in V \otimes^1 = V$$

(Clifford algebra)

Note: $\deg(v w + w v) = 2$, $\deg(2 \omega(v, w)) = 0$,
so not homogeneous relation for $\mathbb{Z}$-grading

of $T(V)$, but is homogeneous for $\mathbb{Z}/2$-grading.

$\Rightarrow$ $C(V) = C^+(V) \otimes C^-(V)$

$\uparrow$ even elements

$\downarrow$ odd elements
C(u) has the following "universal property":

If \(A\) is any associative algebra, then the data of a homomorphism \(\Phi: C(u) \to A\) is the same as a linear map \(\varphi: V \to A\) s.t.

\[
\varphi(v) \cdot \varphi(w) + \varphi(w) \cdot \varphi(v) = 2\omega(v, w) \quad \forall v, w \in V.
\]

Lemma. Let \(v_1, \ldots, v_m\) be a basis of \(V\).

For \(I = (i_1, \ldots, i_k)\) with \(1 \leq i_1 < i_2 < \ldots < i_k \leq m\), define \(v_I = v_{i_1} v_{i_2} \cdots v_{i_k} \in C(u)\). \([v_I] = 1\)

Then \(\{v_I\}\) is a basis for \(C(u)\).

Hence \(\dim C(u) = 2^m\).

Proof. Taking all products of \(v_i\)'s gives basis for \(T(u)\), hence their images in \(C(u)\) span. The Clifford relation \(\Rightarrow\) if \(i > j\), then

\[
V_i V_j = -V_j V_i + 2\omega(v_i, v_j)
\]

can rewrite any product in order.
Furthermore, \( v_i^2 + v_i^2 = 2 \omega(v_i, v_i) \Rightarrow v_i^2 = \omega(v_i, v_i) \). 

\[ \Rightarrow \{ v_I | I = s_1 < \ldots < s_k \leq m \} \text{ span } C(V) . \]

For each \( d \), define \( C \leq d(V) = \text{span} \) of all \( w_1 \ldots w_k \) where \( w_i \in V \) and \( k \leq d \).

\( = \) image of \( V \otimes \cdots \otimes V \otimes 1_{V^*} \in C(V) \) 

\[ \Rightarrow C(V) = \bigoplus_{d \geq 0} C \leq d(V) / C \leq d-1(V) \]

as vector spaces

\[ \text{satisfies the relation for exterior algebra } \Lambda(V) \]

In \( \Lambda(V) \), \( v_i \) linearly independent, so they were linearly independent to begin with. \( \Box \)
Consider $n = 2n$, even
Write $V = W \oplus W'$ where $W, W'$ are both isotropic of dimension $n$.
(If $\{ e_1, \ldots, e_{2n} \}$ is hyperbolic basis, can take
$W = \text{span} \{ e_1, \ldots, e_n \}$, $W' = \text{span} \{ e_{n+1}, \ldots, e_{2n} \}$.)
Note: $W' \cong W \times W$ via $\omega$.

$\Lambda(W) = \bigoplus_{d \geq 0} \Lambda^d(W)$ exterior algebra
(product given by concatenating tensor)

Then if dim $V = 2n$, then $\mathcal{C}(V) \cong \text{End} (\Lambda W)$
i.e., $\mathcal{C}(V) \cong$ algebra of matrices of size $2^n$.
and $\mathcal{C}(V)$ is simple.

**Pf.** First, construct $\Psi : \mathcal{C}(V) \to \text{End} (\Lambda W)$
using universal property of $\mathcal{C}(V)$. 
For \( w \in W \), let \( \varphi(w) \in \text{End}(\wedge^\bullet W) \) be left multiplication by \( w \), i.e.,

\[
\varphi(w)(\alpha) = w \wedge \alpha.
\]

For \( w', w'' \in W \), let \( \varphi(w') \in \text{End}(\wedge^\bullet W) \) be

\[
\varphi(w') = \varphi(w_1 \wedge \ldots \wedge w_d)
\]

\[
2 \sum_{i=1}^d (-1)^{i-1} \wedge(w', w_i) (w_1 \wedge \ldots \wedge \hat{w_i} \ldots \wedge w_d)
\]

For \( w = w + w' \), \( w, w' \in W \), \( \varphi(v) = \varphi(w) + \varphi(w') \)

\( \varphi \) is linear.

Need to check Clifford relations hold.

\[
\varphi(v) \varphi(v') + \varphi(v') \varphi(v) = 2 \wedge(v, v').
\]

Suffices to check when \( v, v' \in W \).

1. If \( v, v' \in W \), then \( \varphi(v) \varphi(v')(\alpha) = vv' \wedge \alpha \)

\[
\varphi(v') \varphi(v)(\alpha) = v' \wedge v \wedge \alpha = -vv' \wedge \alpha
\]

\[\Rightarrow \varphi(v) \varphi(v') + \varphi(v') \varphi(v) = 0 = 2 \wedge(v, v').\]
② If \( v, v' \in W \), \( \varphi(v') \varphi(v) + \varphi(v) \varphi(v') = 0 \)

③ If \( v \in W, v' \in W \), then

\[
\varphi(v) \varphi(v') (w_1, \ldots, w_d) = \sum_{i=1}^{d} (-1)^{i-1} 2 \omega(v', w_i) (w_1, \ldots, \hat{w}_i, \ldots, w_d)
\]

\[
\varphi(v) \sum_{i=1}^{d} (-1)^{i-1} 2 \omega(v, w_i) (v \wedge w_1, \ldots, \hat{w}_i, \ldots, w_d)
\]

\[
\varphi(v') \varphi(v) (w_1, \ldots, w_d) = v \wedge w_1, \ldots, w_d
\]

\[
= 2 \omega(v', v) w_1, \ldots, w_d + \sum_{i=1}^{d} (-1)^{i} 2 \omega(v, w_i) (v \wedge w_1, \ldots, \hat{w}_i, \ldots, w_d)
\]

\[
(\varphi(v) \varphi(v') + \varphi(v') \varphi(v)) (w_1, \ldots, w_d)
\]

\[
= 2 \omega(v', v) (w_1, \ldots, w_d)
\]

\[
\Rightarrow \quad \varphi(v) \varphi(v') + \varphi(v') \varphi(v) = 2 \omega(v', v).
\]

\[
\Rightarrow \quad \text{homomorphism } \varphi : C(V) \to \text{End}(\Lambda^W).
\]
Claim. \( \varphi \) is injective.

Suppose \( a = \sum_{I} \alpha_I e_I \in \ker \varphi \).

Define \( s(I) = \#(I \cap \{n+1, \ldots, 2n\}) \).

We will show by induction on \( s(I) \) that \( \alpha_I = 0 \).

**Base case:** \( s(I) = 0 \Rightarrow I \subseteq \{1, \ldots, n\} \)

\( \varphi(e_I)(1) = 0 \) if \( s(I) > 0 \)

\[ 0 = \varphi(a)(1) = \sum_{I \subseteq \{1, \ldots, n\}} \alpha_I e_I \]

\( \{ e_I \mid I \subseteq \{1, \ldots, n\} \} \) is a basis for \( V(n) \)

\( \Rightarrow \) \( \alpha_I = 0 \) \( \forall I \) s.t. \( s(I) = 0 \).

**Induction step:** assume \( \alpha_I = 0 \) \( \forall I \) s.t. \( s(I) \leq d \), \( \varphi(e_I)(e_{i_1} \land \ldots \land e_{i_{d+1}}) = 0 \) if \( s(I) > d + 1 \)

or if \( s(I) = d + 1 \) and \( I \cap \{n + 1, \ldots, 2n\} \neq \emptyset \) \(2n + 1, -i_{d+1}\}

\( \varphi(e_{\{2n+1-i_1, \ldots, 2n+1-i_{d+1}\}}(e_{i_1} \land \ldots \land e_{i_{d+1}}) = \sum_{I} \alpha_I e_I \)
\[ \Rightarrow \varphi(a)(e_{i_1} \wedge \ldots \wedge e_{i_{d+1}}) \]
\[ = c_{i_1,\ldots,i_{d+1}} \sum_{I \subseteq \{2n+1-i_1,\ldots,2n+1-i_{d+1}\}} \theta_I \] 
\[ \text{subsets of } \{1,\ldots,n\} \]
\[ \Rightarrow \alpha's \text{ appearing are 0.} \text{ (since } e_{i_1} \text{ are basis for } \wedge W) \]

Now vary over all choices of \( \{i_1,\ldots,i_{d+1}\} \) to finish induction.

\[ \Rightarrow \varphi \text{ is surjective.} \]

Since \( \dim C(V) = (2^n)^2 = \dim \text{End}(\wedge W) \),

\( \varphi \) is an isomorphism.

Define \( \wedge W = \bigoplus_{d \geq 0} \wedge^d W \), \( \tilde{\wedge} W = \bigoplus_{d \geq 0} \wedge^{2d+1} W \).

Cor. \( C^+(V) = \text{End}(\wedge W) \times \text{End}(\tilde{\wedge} W) \).

So \( C^+(V) \) is semisimple.
Define \[ \wedge W = \bigoplus_{d \geq 0}^{2d} \wedge^d W, \wedge W = \bigoplus_{d \geq 0}^{2d+1} \wedge^d W. \]

Cor. \[ C^+(V) = \text{End}(\wedge W) \times \text{End}(\wedge W). \]

so \( C^+(V) \) is semisimple.

If. The action of \( C^+(V) \) on \( \wedge W \) preserves both \( \wedge^\text{even} W \) and \( \wedge^\text{odd} W \), so get \( \Phi: C^+(V) \rightarrow \text{End}(\wedge^\text{even} W) \times \text{End}(\wedge^\text{odd} W). \)

Since \( \Phi \) is injective on \( C(V) \), it is also injective on \( C^+(V) \).

\[ \dim C^+(V) = \frac{1}{2} \dim C(V) = 2^{n-1} \]

\[ \dim \wedge W = 2^{n-1}, \quad \dim \text{End}(\wedge^\text{even} W) \times \text{End}(\wedge^\text{odd} W) \]

\[ \dim \wedge W = 2^{n-1} \quad (2^{n-1})^2 + (2^{n-1})^2 = 2^{2n-1}. \]

\[ \implies \dim C^+(V) = \dim(\text{End}(\wedge^\text{even} W) \times \text{End}(\wedge^\text{odd} W)) \]

\[ \implies \Phi \text{ is an isom.} \]
Consider \( \dim V = 2n+1 \) odd.
Write \( V = W \oplus W' \oplus L \), \( W, W' = n \)-dim isotropic spaces, \( L = (W \oplus W')^\perp \).

(If \( \{e_1, \ldots, e_{2n+1}\} \) hyperbolic basis, then
\[ W = \text{span} (e_1, \ldots, e_n), \quad L = \text{span} (e_{n+1}) \]
\[ W' = \text{span} (e_{n+2}, \ldots, e_{2n+1}) \].

Then \( \dim V \text{ odd } \Rightarrow C(U) \cong \text{End} (\Lambda W) \times \text{End} (\Lambda W') \).
Hence \( C(U) \cong \text{product of } 2 \text{ matrix algebra of size } 2^n \text{ each, } \)
\( C(U) \) semisimple.

Furthermore, \( C^+ (U) \cong \text{End} (\Lambda W) \cong \text{End} (\Lambda W') \).

If \( \Phi : C(U) \to \text{End} (\Lambda W) \) using universal property. If \( v \in W \oplus W' \), \( \Phi (v) \) as before.
If \( l \in L \) and \( \langle l, l \rangle = 1 \), define (for \( \lambda \in \mathbb{C} \))
\[ \Phi (l) \left( w, \ldots, \ldots, w, d \right) = \lambda \langle l, w \rangle w, \ldots, w, d. \]
Clifford relations:

1. \( \Psi(\lambda e) \Psi(\mu e)(w, \ldots, w_d) = \Psi \mu w, \ldots, w_d \Rightarrow \Psi(\lambda e) \Psi(\mu e) + \Psi(\mu e) \Psi(\lambda e) = 2\mu e = 2w(\lambda e, \mu e) \)

2. If we work, note that

   \( \Psi(w) \) changes parity of an element

   \( \Psi(w) \Psi(\lambda e) + \Psi(\lambda e) \Psi(w) = 0 = 2w(w, \lambda e) \Rightarrow \) get homomorphism \( \Psi : C(V) \to \text{End}(\wedge W) \).

   Also need \( \Psi' : C(V) \to \text{End}(\wedge W') \), defined similarly: reverse roles of \( W, W' \), \( \Psi(\lambda) \) modified by \( (-1)^n \) (omit check)

   \( \Rightarrow \) homomorphism \( \Phi : C(V) \to \text{End}(\wedge W) \times \text{End}(\wedge W') \).

   Need to check \( \Phi \) injective. Use idea similar to even case. (Omit details)
Now consider action of $C^\infty(V)$ on $\Lambda W$ via $\Phi$. Need to show $\Phi: C^\infty(V) \rightarrow \operatorname{End}(\Lambda W)$ is injective. Argument very similar to even case. (omit details).

**Lemma.**
1. If $x \in C^\infty(V)$ and $xv = vx$ for all $v \in V$, then $x \in C^0(V)$ is a scalar.
2. If $x \in C^-(V)$, and $xv = -vx$ for all $v \in V$, then $x = 0$. 