**Math 251C, Lecture 4**

**Multiplicity-Free Actions**

Ref: Howe, "Perspectives on invariant theory"

Last time: irreducible reps of $GL_n \mathbb{C}$

$$S^\lambda \mathbb{C}^n$$

$$\lambda \geq \ldots \geq \lambda_n \in \mathbb{Z}^n$$

$W$ rep $\Rightarrow$ $W \cong \bigoplus (S^\lambda \mathbb{C}^n)^{\otimes m, \lambda}$ $\leftarrow$ multiplicity

Def. $W$ is **multiplicity-free** if $m_{\lambda} \leq 1 \ \forall \ \lambda$

Similarly, if $W$ is rep of $GL_n \mathbb{C} \times GL_m \mathbb{C}$,

then $W \cong \bigoplus (S^\lambda \mathbb{C}^n \otimes S^\mu \mathbb{C}^m)^{\otimes m, \lambda, \mu}$

$W$ multiplicity-free if $m_{\lambda, \mu} \leq 1 \ \forall \ \lambda, \mu$

Note: $m_\lambda = \dim$ of space of h.w. vectors in $W$ at weight $\lambda$
Given vector space $U$, let $\text{Sym}^d U^*$

$= d^{th}$ symmetric power of $U^*$

$= (U^*)^d / \left< u_1 \otimes \cdots \otimes u_d - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \right>_{\sigma \in S_d, \ u_1, \ldots, \ u_d \in U^*}$

symmetric group

If we pick basis $x_1, \ldots, x_n$ of $U^*$, then $\text{Sym}^d U^*$ is the space of homogeneous degree $d$ polynomials in $x_1, \ldots, x_n$.

$\text{Sym} U^* := \bigoplus_{d \geq 0} \text{Sym}^d U^*$

$= \text{all polynomials in } x_1, \ldots, x_n$.

Note: If $\text{GL}_k$ acts on $U$, then $\text{Sym}^d U^*$ & also $\text{Sym} U^*$ are representations of $\text{GL}_k$.
If $f \in \text{Sym} U^*$, $u \in U$, then evaluation $f(u) \in C$ makes sense.

$\text{Sym } U^*$ is infinite-dim. $\mathbb{Q}$, but each $\text{Sym}^d U^*$ is finite-dim. $\mathbb{Q}$, so previous results apply.

**Example** $G\text{-}L(V)$ acting on $U = V^*$, so $U^\perp = V$

Claim: $\text{Sym}^d V$ is irreducible for all $d \geq 0$.

Pick a basis $x_1, \ldots, x_n$ for $V$.

Weight vectors of $\text{Sym}^d V$ are monomials in $x_1, \ldots, x_n$. $x_1^{d_1} \cdots x_n^{d_n} \mapsto$ weight $(d_1, \ldots, d_n)$

$x_1^d$ is h.w. and only one, weight is $(d, 0, \ldots, 0)$

Eg. $(a \ b) \cdot x_1 x_2 = (ax_1)(bx_1 + cx_2)$

$\implies \text{Sym } V$ is multiplicity-free.

**Goal**: Find condition on $U$ so that $\text{Sym } U^*$ is multiplicity-free.
Zariski topology

\( U = \text{vector space, } f: U \to \mathbb{C} \) is polynomial if it is so wrt a basis for \( U \).

Given a set of polynomials \( f_i: U \to \mathbb{C} \)

\[ Z(I) := \{ u \in U \mid f_i(u) = 0 \ \forall f_i \in I \} \]

(zero set of \( I \))

WLOG, we usually assume that \( I \) is an ideal:

1. \( f + g \in I \) \( \forall f, g \in I \)
2. \( fh \in I \) \( \forall f \in I, h \text{ arbitrary polynomial} \)
3. \( I \neq \emptyset \).

Given a set of polynomials \( \{ f_i \} \),

\( \langle f_i \rangle = \text{smallest ideal containing all } f_i \)

\[ = \{ h_i f_1 + \ldots + h_r f_r \mid h_i \text{ arbitrary} \} \]

\( I \) is generated by \( \{ f_i \} \) if \( I = \langle f_i \rangle \).

Thm (Hilbert basis thm) Every ideal can be generated by finite set of polynomials.
Sum of ideals: \( I_1 + I_2 = \{ f + g \mid f \in I_1, g \in I_2 \} \)

Product of ideals: \( I_1 I_2 = \langle fg \mid f \in I_1, g \in I_2 \rangle \)

(Infinitesimal sums of ideals ok)

**Def.** The Zariski topology on \( U \) is the topology whose closed sets are the \( \mathbb{Z}(I) \).

**Check:**
1. \( \emptyset = \mathbb{Z}(\langle 1 \rangle) \)
2. \( U = \mathbb{Z}(\langle 0 \rangle) \)
3. Intersection: \( \bigwedge_j \mathbb{Z}(I_j) = \mathbb{Z}\left( \sum_j I_j \right) \)
4. Finite unions: \( \mathbb{Z}(I_1) \cup \ldots \cup \mathbb{Z}(I_r) = \mathbb{Z}(I_1 \cdots I_r) \).

The \( \mathbb{Z}(I) \) are affine varieties.

Given an affine variety \( X \subset \mathbb{C}^n \), denote \( \mathbb{C}[X] \) the quotient of \( \text{Sym} \, X^* \) by ideal of all polynomials which are identically zero on \( X \).

**Note:** \( \mathbb{C}[U] = \text{Sym} \, U^* \)
Ex. $\text{GL}_n \mathbb{C} \subset \mathbb{C}^{n^2}$ is complement of affine variety $\mathbb{Z}(\langle \text{det} \rangle) \Rightarrow \text{GL}_n \mathbb{C}$ is open in Zariski topology

Introduce new variable $t$

Consider $\mathbb{Z}(\langle t \cdot \text{det} - 1 \rangle) \subset \mathbb{C}^{n^2+1}$

\[
\{ (g, x) \mid \lambda \cdot \text{det}(g) = 1 \} = \{ (g, x) \mid \lambda = \frac{1}{\text{det}g} \}
\]

$\lambda$ is redundant

Projecting onto first $n^2$ coordinates, we get a bijection $\mathbb{Z}(\langle t \cdot \text{det} - 1 \rangle) \sim \mathbb{C}^{n^2} \setminus \mathbb{Z}(\langle \text{det} \rangle) 

\Rightarrow \text{GL}_n \mathbb{C}$ is an affine variety.

In fact, an algebraic group.

**Det** A topological space $X$ is irreducible if, whenever $X = X_1 \cup X_2$, $X_i$ closed subsets, then $X = X_1$ or $X = X_2$. 
Prop. A vector space $V$ with Zariski topology is irreducible.

Pf. Suppose $V = \mathbb{Z}(I_1) \cup \mathbb{Z}(I_2) = \mathbb{Z}(I_1, I_2)$ for ideals $I_1, I_2$. Then $I_1 I_2 = 0$ since every nonzero polynomial has a $u \in U$ such $\exists f(u) \neq 0$.

$\Rightarrow$ $fg = 0 \ \forall f \in I_1, g \in I_2$

$\Rightarrow$ If $I_1 \neq 0$, then $I_2 = 0$ since $fg \neq 0$ whenever $f \neq 0$ and $g \neq 0$.

(1) If $I_1 \neq 0$, then $I_2 = 0$ since $fg \neq 0$ whenever $f \neq 0$ and $g \neq 0$.

(2) Similarly, if $I_2 \neq 0$, then $I_1 = 0$.

In case 1, $\mathbb{Z}(I_1) = U$ ✓

In case 2, $\mathbb{Z}(I_1) = U$ ✓

Else, both $I_1 = I_2 = 0$ ✓.

Prop. Every nonempty open subset $Y$ of an irreducible space is dense (i.e., if $X' \supseteq Y$ is closed, then $X' = X$)

Pf. $X = X' \cup (X \setminus Y)$, both closed.

$X_{\text{irred}} \Rightarrow X = X'$ ✓

or $X = X \setminus Y \Rightarrow Y$ is empty ✓.
Now consider $U$ is $GL(V)$-rep, $X \subset U$ affine variety. $G < GL(V)$ subgroup s.t. $X$ is closed under action of $G$.

The $G$-orbits of $X$ are the equivalence classes of the relation $x \sim x'$ if $\exists g \in G$ s.t. $x' = gx$.

Thus let $X$ be an affine variety in a rep of $GL(V)$. Let $B < GL(V)$ Borel subgroup.

Assume: $\exists B$-orbit $Y \subset X$ which is dense. Then

(a) $C[X]$ is multiplicity-free rep.

(b) Let $\lambda$ be a h.w. of h.w. vector in $C[X]$.

Pick $u \in Y$ and let $H = \text{stab}(u) = \{g \in GL(V) ~|~ h.u = u\}$

Then, $\lambda(h) = 1 \quad \forall h \in H \cap B$. 
