Math 251C, Lecture 8

Last time: \( S_\lambda V \) has basis of weight vectors indexed by semistandard Young tableaux (SSYT).

Formula for \( \# \text{SSYT} \):

1. \( \dim S_\lambda \mathbb{C}^n = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \)

2. (Hook-content formula) \( \dim S_\lambda \mathbb{C}^n = \prod_{u \in Y(\lambda)} \frac{n + c(u)}{h(u)} \)

where \( c(u) = j - i \) where \( j = \) column index of \( u \) \( i = \) row index of \( u \)

\( h(u) \) = \# boxes in same row of \( u \) appearing to the right of \( u \) + \# boxes in same column of \( u \) appearing below \( u \) + 1

\( \dim S_\lambda \mathbb{C}^n \) is a polynomial in \( n \) of degree \( |\lambda| \).
\[ \lambda = (6, 3, 1) \]

\[ \text{hook length} \]

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
-1 & 0 & 1 & 4 \\
-2 & \text{ } & \text{ } & 5 \\
\end{array}
\]

\[ \lambda = (3, 2) \Rightarrow \dim S_{32} \sigma^n \]

1. \[ i = 1: \quad \frac{3 - 2 + 1}{1} \cdot \frac{3 + 2}{2} \cdot \frac{3 + 3}{3} \cdots \frac{3 + (n-1)}{n-1} = 2 \cdot \frac{(n+2)!}{4!} \cdot \frac{1}{(n-1)!} \]

2. \[ i = 2: \quad \frac{2 + 1}{1} \cdot \frac{2 + 2}{2} \cdot \frac{2 + (n-2)}{n-2} = \frac{n!}{2(n-2)!} \]

\[ \dim S_{32} \sigma^n = 2 \cdot \frac{(n+2)!}{4!(n-1)!} \cdot \frac{n!}{2(n-2)!} = \frac{n(n+2)(n+1)(n)(n-1)}{4!} \]

\[ \text{polynomial in } n \text{ of degree 5} \]

2. \[ \frac{c}{h} \]

\[
\begin{array}{cccc}
6 & 1 & 2 & 0 \\
1 & 4 & 3 & 1 \\
0 & 2 & 1 & \text{ } \\
\end{array}
\]

\[ \dim S_{32} \sigma^n = \frac{n(n+1)(n+2)(n-1)n}{4 \cdot 3 \cdot 2 \cdot 1} \]

\[ = \frac{n^2(n+1)(n+2)(n-1)}{4!} \]
Symmetric polynomials/ functions

Lemma. Let $\rho$ be rep of $\text{GL}_n \mathbb{C}$. Then

$(\text{char } \rho)(x_1, \ldots, x_n)$ is symmetric, i.e., $\forall \sigma \in S_n$

$(\text{char } \rho)(x_1, \ldots, x_n) = (\text{char } \rho)(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

Pf. $\sigma \mapsto$ permutation matrix $M(\sigma)$

(corresponding to linear map $e_i \mapsto e_{\sigma(i)}$)

$M(\sigma)^{-1} \begin{pmatrix} x_1 & 0 \\ 0 & x_n \end{pmatrix} M(\sigma) = \begin{pmatrix} x_{\sigma(1)} & \cdots \\ \cdots & x_{\sigma(n)} \end{pmatrix}$

$(\text{char } \rho)(x_1, \ldots, x_n) = \text{trace } \rho(\begin{pmatrix} x_1 & \cdots \\ \cdots & x_n \end{pmatrix})$

& trace is invariant under conjugation. □

Recall: for every rep $\rho$, $\exists d$ s.t. $\rho \otimes \text{det}^d$

is polynomial. Character of polynomial rep is a polynomial in $x_1, \ldots, x_n$.

$\implies \text{char } \rho(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{(x_1, \ldots, x_n)^d}$

$\text{f is a polynomial}$
Let $\Lambda(n)$ = set of symmetric polynomials in $x_1, \ldots, x_n$ w/ $\mathbb{Z}$-coefficients. $\Lambda(n)$ is a ring under usual addition/multiplication which contains charp for any polynomial rep $\rho$.

For polynomial reps of $GL_n \times GL_m \mathbb{C}$, let $\Lambda(n, m)$ = polynomials w/ $\mathbb{Z}$-coefficients in $x_1, \ldots, x_n, y_1, \ldots, y_m$ which are symmetric in $x$'s & $y$'s.

$\Lambda(n, m)$ contains characters of polynomial reps of $GL_n \times GL_m \mathbb{C}$.

**Def.** The Schur polynomial $s_{\lambda}(x_1, \ldots, x_n)$ is character of $S_{\lambda} \mathbb{C}^n$. $s_{\lambda}(x_1, \ldots, x_n) = \sum x^\mu \tau^{\mu}$, $SSYT \tau$ of shaped $\lambda$.

**General facts**

1. $\{s_{\lambda}(x) \mid \lambda(\lambda) \leq n\}$ is a basis for $\Lambda(n)$.
(2) Weyl character formula

Given $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$, define

$$a_{\alpha} = \det(x_i^j), \quad i,j = 1, \ldots, n = \det \left( \begin{array}{cccc} x_1^{a_1} & x_1^{a_2} & \cdots & x_1^{a_n} \\ x_2^{a_1} & x_2^{a_2} & \cdots & x_2^{a_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{a_1} & x_n^{a_2} & \cdots & x_n^{a_n} \end{array} \right)$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \times \sigma(\alpha)$$

Define $\rho = (n-1, n-2, \ldots, 1, 0) \in \mathbb{Z}_{\geq 0}^n$

**Thm. (WCF)**

$$s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho} = \frac{\det(x_i^{j+n-j})_{i,j=1,\ldots,n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}$$

$$= \sum_{\rho \in S_n} \text{sgn}(\sigma) \times \sigma(\lambda+\rho)$$

$$= \frac{\sum_{\rho \in S_n} \text{sgn}(\sigma) \times \sigma(\rho)}{\sum_{\rho \in S_n} \text{sgn}(\sigma) \times \sigma(\rho)}$$
Plethysm

Given a rep $W$ of $GL_n \mathbb{C}$, we can construct $S^2(W)$ and think of it as a rep of $GL_n \mathbb{C}$

Let $w_1, \ldots, w_N$ be weights of $W$ (repeating if appear \( w \) with multiplicity)

\[
(\text{char } S^2 W)(x_1, \ldots, x_n) = S^2(x^{w_1}, x^{w_2}, \ldots, x^{w_N})
\]

We can also define for $\text{char } W$

\[
S^2 \circ \text{char } W
\]

for any symmetric polynomial $f$ in $N$ vars:

\[
f = \sum \lambda \; c_{\lambda} s_{\lambda}; \quad \text{for char } W = \sum \lambda \; c_{\lambda} (S^2 \circ \text{char } W)
\]

Question: given $\lambda, \mu$ we can define $S^\lambda(S^\mu(C^n))$. How does it decompose into irreducible reps? $\cong \bigoplus (S^\nu(C^n)) \otimes P^\nu_{\lambda \mu}$

Very hard, outside special cases.
Introduce new variable \( t \), we will work w/ power series in \( t \) w/ coefficients in \( \Lambda(n) \).

\[
hd(x) = \text{char } \text{Sym}^d \mathbb{C}^n = s_d(x)
\]

Lemma \( \sum_{n \in \mathbb{N}} \sum_{d \geq 0} h_d(x_1, \ldots, x_n) t^d = \frac{1}{\prod_{i=1}^{n} (1 - x_i t)} \)

Pf. LHS is sum of \( x_1^{d_1} \cdots x_n^{d_n} t^{d_1 + \cdots + d_n} \)

varying over all choices of \( d_i \geq 0 \)

RHS, using geometric series is \( \prod_{i=1}^{n} (\sum_{d_i \geq 0} x_i^d t^{d_i}) \)

which is the same \( \square \)

\textbf{Thm} (Cauchy identity) in \( \Lambda(n,m) \mathbb{C}[t] \):

\[
\prod_{i=1}^{n} \prod_{j=1}^{m} (1 - x_i y_j t)^{-1} = \sum_{\lambda} s_{\lambda} (x_1, \ldots, x_n) s_{\lambda}(y_1, \ldots, y_m) t^{\rho}
\]

sum over all \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r \geq 0) \) \( r = \min(n, m) \)