The topic of the course is representations of Lie groups, with an emphasis on classical (general linear, symplectic, orthogonal) groups. We will explore this from several different angles: linear algebra, invariant theory, combinatorics, algebraic geometry. The perspective I will take is that the representation theory of general linear groups is a natural extension of linear algebra and for symplectic and orthogonal groups, we are doing linear algebra in the presence of a (skew-)symmetric bilinear form. I intend to take advantage of theorems from the general theory (I will recall the necessary background as we go), but our focus will be to explore examples and explicit constructions rather than the proofs of such theorems.

I’m not using any particular reference, but [FH] comes the closest to what I want to cover. For simplicity, we’ll work with groups over the complex numbers. Some of what we say will work over different fields. Another disclaimer: I don’t intend for this to be a comprehensive reference. So results will usually be stated in special cases the first time even if a more general case is needed later. Instead, I will try to minimize technical definitions for later cases until they are needed, and then restate results in more general forms.

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1. General linear group

Let \( V \) be a finite-dimensional complex vector space. We let \( \text{GL}(V) \) denote the group of invertible linear operators \( V \to V \) where the group operation is composition. If \( V = \mathbb{C}^n \), then we write \( \text{GL}_n(\mathbb{C}) \) instead of \( \text{GL}(V) \). The only difference is that \( \mathbb{C}^n \) has a chosen basis (the standard one \( e_1, \ldots, e_n \)) while \( V \) does not. This is a Lie group (in fact, a complex Lie group).
1.1. **Representations.** Some basic notation we will use over and over again with regards to $\text{GL}_n(\mathbb{C})$: we let $B \subset \text{GL}_n(\mathbb{C})$ be the subgroup of upper-triangular matrices (called a **Borel subgroup**) and $T \subset \text{GL}_n(\mathbb{C})$ be the subgroup of diagonal matrices (called a **maximal torus**). When talking about $\text{GL}(V)$, a subgroup $B$ is a Borel subgroup if it is the group of upper-triangular matrices with respect to some choice of basis for $V$. Similarly, $T$ is a maximal torus if it is the group of diagonal matrices with respect to some choice of basis for $V$. So note that $T$ and $B$ are not unique, but choosing a basis will determine a choice (and this will be convenient when we want to do calculations).

An **algebraic representation** (or **rational representation**) of $\text{GL}(V)$ is a group homomorphism $\rho: \text{GL}(V) \rightarrow \text{GL}(W)$ for some other finite-dimensional complex vector space $W$ which is algebraic: this means that for some (equivalently, any) choice of ordered bases for $V$ and $W$, then for any $g \in \text{GL}(V)$ the entries of $\rho(g)$ are all rational functions of the entries of $g$. All representations that we consider will be algebraic, so we will just say representation. If the entries are all polynomial functions, then we call $\rho$ a **polynomial representation**.

**Example 1.1.1.**

- **If** $V = W$ we can take $\rho$ to be the identity.
- **If** $V = W = \mathbb{C}^n$, take $\rho(g) = (g^{-1})^T$ where $T$ is transpose. This is not a polynomial representation.
- Take $V = \mathbb{C}^2$ and $W = \mathbb{C}^3$ and $\rho$ given by
  $$\text{GL}_2(\mathbb{C}) \rightarrow \text{GL}_3(\mathbb{C})$$
  $$\left(\begin{array}{cc} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{array} \right) \mapsto \left(\begin{array}{ccc} g_{1,1}^2 & g_{1,1}g_{1,2} & g_{1,2}^2 \\ 2g_{1,1}g_{2,1} & g_{1,1}g_{2,2} + g_{1,2}g_{2,1} & 2g_{1,2}g_{2,2} \\ g_{2,1}^2 & g_{2,1}g_{2,2} & g_{2,2}^2 \end{array} \right)$$

- Take $V = \mathbb{C}^n$ and $W = \mathbb{C}$ and $\rho(g) = \det(g)$. More generally, for any integer $d$, we can take $\rho(g) = \det(g)^d$. This is a polynomial representation if and only if $d \geq 0$.
- For a non-example, take $V = W = \mathbb{C}^n$. Take $\rho(g) = \overline{g}$ where $\overline{g}$ means take the complex conjugate of each entry. Then $\rho: \text{GL}_n(\mathbb{C}) \rightarrow \text{GL}_n(\mathbb{C})$ is a homomorphism, but it is not algebraic.

We will think of $\rho$ as giving us an action of $\text{GL}(V)$ on $W$, i.e., if $g \in \text{GL}(V)$ and $w \in W$, we define $g \cdot w = \rho(g)(w)$. Then this is a group action in the usual sense. Sometimes we will call $W$ the representation.

Here are some general facts. First, pick a maximal torus $T \subset \text{GL}(V)$. If $\rho: \text{GL}(V) \rightarrow \text{GL}(W)$ is a representation, then there is a basis $w_1, \ldots, w_r$ for $W$ such that each $w_i$ is an eigenvector of $\rho(t)$ for all $t \in T$. Pick an ordered basis for $V$ so that $T$ is the set of diagonal matrices. If the entries of $t$ are $x_1, \ldots, x_n$, then with respect to the basis $w_1, \ldots, w_r$, $\rho(t)$ is diagonal and the entries are of the form $x_1^{\mu_1} \cdots x_n^{\mu_n}$ for some $\mu_1, \ldots, \mu_n \in \mathbb{Z}$ (why?). Any vector $w$ which is an eigenvector for all $\rho(t)$ is called a **weight vector** and $(\mu_1, \ldots, \mu_n)$ is its **weight** if $\rho(t)(w) = x_1^{\mu_1} \cdots x_n^{\mu_n}w$ for all $x_1, \ldots, x_n$. We also write $\mu(t)$ for $x_1^{\mu_1} \cdots x_n^{\mu_n}$.

The **character** of $\rho$ is defined to be the function $\text{char}(\rho)(x_1, \ldots, x_n) = \text{Tr} \rho(t)$ where $t$ is the diagonal matrix with entries $x_1, \ldots, x_n$ and $\text{Tr}$ denotes trace. Alternatively, $\text{char}(\rho)(x_1, \ldots, x_n) = \sum_{i=1}^{\infty} x_1^{\mu_{1,i}} \cdots x_n^{\mu_{n,i}}$.

**Example 1.1.2.** We compute the characters from Example 1.1.1.

- The basis $w_1, \ldots, w_n$ is already an eigenbasis, so the character is $x_1 + \cdots + x_n$.
- Again, the standard basis is an eigenbasis, so the character is $x_1^{-1} + \cdots + x_n^{-1}$.
• Taking \( g \) to be diagonal with entries \( x_1, x_2 \), we see that \( \rho(g) \) is also diagonal and its trace is \( x_1^2 + x_1 x_2 + x_2^2 \).
• The character is \((x_1 \cdots x_n)^d\).

Basic operations transform easily on the level of characters:

- If \( \rho_i : \text{GL}_n(C) \to \text{GL}(V_i) \) are representations for \( i = 1, 2 \), we can form the direct sum representation \( \rho_1 \oplus \rho_2 : \text{GL}_n(C) \to \text{GL}(V_1 \oplus V_2) \) via
  \[
  (\rho_1 \oplus \rho_2)(g) = \begin{pmatrix}
  \rho_1(g) & 0 \\
  0 & \rho_2(g)
\end{pmatrix}
\]
  and
  \[
  \text{char}(\rho_1 \oplus \rho_2)(x_1, \ldots, x_n) = \text{char}(\rho_1)(x_1, \ldots, x_n) + \text{char}(\rho_2)(x_1, \ldots, x_n).
\]
  In terms of group action, this is given by \( g \cdot (v_1, v_2) = (g \cdot v_1, g \cdot v_2) \).

- If \( \rho_i : \text{GL}_n(C) \to \text{GL}(V_i) \) are representations for \( i = 1, 2 \), we can form the tensor product representation \( \rho_1 \otimes \rho_2 : \text{GL}_n(C) \to \text{GL}(V_1 \otimes V_2) \) via (assuming \( \rho_1(g) \) is \( N \times N \)):
  \[
  (\rho_1 \otimes \rho_2)(g) = \begin{pmatrix}
  \rho_1(g)_1,1 \rho_2(g) & \rho_1(g)_1,2 \rho_2(g) & \cdots & \rho_1(g)_1,N \rho_2(g) \\
  \rho_1(g)_2,1 \rho_2(g) & \rho_1(g)_2,2 \rho_2(g) & \cdots & \rho_1(g)_2,N \rho_2(g) \\
  \vdots & \vdots & \ddots & \vdots \\
  \rho_1(g)_N,1 \rho_2(g) & \rho_1(g)_N,2 \rho_2(g) & \cdots & \rho_1(g)_N,N \rho_2(g)
\end{pmatrix}
\]
  (here we are multiplying \( \rho_2(g) \) by each entry of \( \rho_1(g) \) and creating a giant block matrix)
  and
  \[
  \text{char}(\rho_1 \otimes \rho_2)(x_1, \ldots, x_n) = \text{char}(\rho_1)(x_1, \ldots, x_n) \cdot \text{char}(\rho_2)(x_1, \ldots, x_n).
\]
  In terms of group action, this is given by \( g \cdot \sum_i v_i \otimes w_i = \sum_i g \cdot v_i \otimes g \cdot w_i) \).

- If \( \rho : \text{GL}_n(C) \to \text{GL}(V) \) is a representation, then we have an action on the dual space \( V^* \) as follows. Given a linear functional \( f : V \to C \), we define \( g \cdot f \) to be the linear functional given by \( (g \cdot f)(v) = f(g^{-1} \cdot v) \). This gives a representation \( \rho^* : \text{GL}_n(C) \to \text{GL}(V^*) \). In terms of matrices, if we pick an ordered basis for \( V \) and use the dual basis for \( V^* \), we have \( \rho^*(g) = (\rho(g)^{-1})^T \) where \( T \) denotes transpose.
  The character is given by \( \text{char}(\rho^*)(x_1, \ldots, x_n) = \text{char}(\rho)(x_1^{-1}, \ldots, x_n^{-1}) \).

Given two representations \( W \) and \( W' \) of \( \text{GL}(V) \), a **homomorphism** between them is a linear map \( f : W \to W' \) such that \( f(g \cdot w) = g \cdot f(w) \) for all \( g \in \text{GL}(V) \) and \( w \in W \). We say that \( W \) and \( W' \) are **isomorphic** if there is a homomorphism between them which is an invertible linear map.

A **subrepresentation** of \( W \) is a subspace \( U \subseteq W \) such that \( g \cdot u \in U \) for all \( u \in U \) and \( g \in \text{GL}(V) \). A nonzero representation \( W \) is **irreducible** (or **simple**) if its only subrepresentations are either 0 or \( W \). A representation is **semisimple** if it is isomorphic to a direct sum of simple representations.

**Theorem 1.1.3.**

1. Every finite-dimensional representation of \( \text{GL}(V) \) is semisimple.
2. Two representations of \( \text{GL}(V) \) are isomorphic if and only if they have the same character.
3. (Schur’s lemma) There are no nonzero homomorphisms between non-isomorphic simple representations. Any homomorphism from a simple representation to itself must be a scalar multiple of the identity.
Let \( B \subset \text{GL}(V) \) be a Borel subgroup which contains our maximal torus \( T \). A nonzero vector \( w \in W \) is a **highest weight vector** \( b \cdot w \) is a scalar multiple of \( w \) for all \( b \in B \). In particular, it is an eigenvector for all \( \rho(t) \) for \( t \in T \), so it is also a weight vector. If \( \mu \) is the weight of this vector, we write \( \mu(b) \) for the scalar multiple i.e., \( b \cdot w = \mu(b)w \).

**Example 1.1.4.** We discuss the examples from Example 1.1.1.

- The standard basis vectors \( w_1, \ldots, w_n \) are the weight vectors. The weight of \( w_i \) is \((0, \ldots, 1, \ldots, 0)\) with a 1 in position \( i \). The only highest weight vector is \( w_1 \) with weight \((1,0,\ldots,0)\).
- The standard basis vectors \( w_1, \ldots, w_n \) are the weight vectors. The weight of \( w_i \) is \((0, \ldots, -1, \ldots, 0)\) with a \(-1\) in position \( i \). The only highest weight vector is \( w_n \) with weight \((0, \ldots, 0, -1)\).
- The standard basis vectors \( e_1, e_2, e_3 \) of \( \mathbb{C}^3 \) are weight vectors with weights \((2,0),(1,1),(0,2)\). The only highest weight vector is \( e_1 \) with weight \((2,0)\).
- \( 1 \in \mathbb{C} \) is a weight vector with weight \((d, \ldots, d)\) and is also a highest weight vector. \( \Box \)

**Theorem 1.1.5.**

- Every finite-dimensional representation contains a highest weight vector.
- Any two highest weight vectors (for a particular choice of Borel subgroup) in an irreducible representation are scalar multiples of each other.
- The weight \((\mu_1, \ldots, \mu_n)\) of a highest weight vector satisfies \( \mu_1 \geq \cdots \geq \mu_n \).
- For every \((\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n\) satisfying \( \mu_1 \geq \cdots \geq \mu_n \), there is an irreducible representation whose highest weight vector has that weight. Furthermore, this representation is unique up to isomorphism and the representation is polynomial if and only if \( \mu_n \geq 0 \).

In particular, we conclude that the irreducible representations are determined by their highest weight vectors, and if we want to express an arbitrary representation as a direct sum of irreducible representations, it suffices to find all highest weight vectors (why?). We use \( S_\lambda(V) \) to denote an irreducible representation whose highest weight is \( \lambda \). We will see different ways to construct this representation.

Note that tensoring a representation by the \( d \)th power of the determinant representation adds \( d \) to all components of each weight, and in particular for highest weights. So the result above implies the following, for which we will give an independent proof:

**Proposition 1.1.6.** If \( W \) is a rational representation, then there exists \( d \) such that \( W \otimes \det^d \) is a polynomial representation.

**Proof.** With respect to some choice of basis, the entries of \( \rho : \text{GL}_n(\mathbb{C}) \to \text{GL}(W) \) are rational functions \( a(x)/b(x) \) of the matrix entries \( x_{i,j} \), such that \( b(g) \neq 0 \) whenever \( g \) is invertible, so it suffices to prove that for every such rational function, \( b \) is a power of the determinant function (up to a scalar). We prove this by induction on \( \deg b \). If \( \deg b = 0 \), then it is a constant and there is nothing to say.

Otherwise, since \( \mathbb{C} \) is algebraically closed, the polynomial \( b \) has a zero somewhere on the space of all matrices. By assumption, all of its zeros are singular matrices, so that \( b(x) = 0 \) implies that \( \det(x) = 0 \). The next result shows that \( \det \) is an irreducible polynomial, so that \( \det \) divides \( b \). But then \( b' = b/\det \) is another polynomial of lower degree such that \( b'(g) \neq 0 \) whenever \( g \) is invertible, so by induction, \( b' \) is a scalar times a power of the determinant. \( \Box \)

**Lemma 1.1.7.** As a polynomial in the \( n \) variables \( x_{i,j} \), \( \det \) is irreducible.
Proof. Suppose we have a factorization \( \det = \alpha \beta \). Note that \( \det \) has degree 1 in each variable \( x_{i,j} \) separately, so for each \( x_{i,j} \), it must be that either \( \alpha \) has degree 1 and \( \beta \) has degree 0 with respect to \( x_{i,j} \), or the other way around. Now consider \( x_{i,j} \) and \( x_{i',j} \) together. Since no term of \( \det \) involves both at the same time, if \( \alpha \) has degree 1 in \( x_{i,j} \), then it must also have degree 1 in \( x_{i',j} \) (if not, then we can write \( \alpha = \alpha_0 x_{i,j} + \alpha_1 \) and \( \beta = \beta_0 x_{i',j} + \beta_1 \) where none of \( \alpha_0, \alpha_1, \beta_0, \beta_1 \) involve \( x_{i,j} \) or \( x_{i',j} \) and multiplying it out gives a contradiction). The same is true if we consider two variables \( x_{i,j} \) and \( x_{i,j'} \) together. This implies that if \( \alpha \) has degree 1 in one of the variables, then it has degree 1 in all of the variables, i.e., \( \beta \) is a constant. Otherwise, the same reasoning implies that \( \beta \) has degree 1 in all of the variables and \( \alpha \) is a constant. \( \square \)

1.1.1. Partitions. A partition of a nonnegative integer \( n \) is a sequence \( \lambda = (\lambda_1, \ldots, \lambda_k) \) such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \) and \( \lambda_1 + \cdots + \lambda_k = n \). We will consider two partitions the same if their nonzero entries are the same. It will also be convenient to make the convention that \( \lambda_i = 0 \) whenever \( i > \ell(\lambda) \). And for shorthand, we may omit the commas, so the partition \( (1,1,1,1) \) of 4 can be written as 1111. As a further shorthand, the exponential notation is used for repetition, so for example, \( 1^4 \) is the partition \( (1,1,1,1) \). We let \( \text{Par}(n) \) be the set of partitions of \( n \), and denote the size by \( p(n) = |\text{Par}(n)| \). By convention, \( \text{Par}(0) \) consists of exactly one partition, the empty one.

Example 1.1.8.

\[
\text{Par}(1) = \{1\}, \\
\text{Par}(2) = \{2, 1^2\}, \\
\text{Par}(3) = \{3, 21, 1^3\}, \\
\text{Par}(4) = \{4, 31, 22, 21^2, 1^4\}, \\
\text{Par}(5) = \{5, 41, 32, 31^2, 2^21, 21^3, 1^5\}.
\]

If \( \lambda \) is a partition of \( n \), we write \( |\lambda| = n \) (size). Also, \( \ell(\lambda) \) is the number of nonzero entries of \( \lambda \) (length). For each \( i \), \( m_i(\lambda) \) is the number of entries of \( \lambda \) that are equal to \( i \).

It will often be convenient to represent partitions graphically. This is done via Young diagrams \( \hat{Y}(\lambda) \), which is a collection of left-justified boxes with \( \lambda_i \) boxes in row \( i \). For example, the Young diagram

![Young diagram](image)

corresponds to the partition \( (5, 3, 2) \). Flipping across the main diagonal gives another partition \( \lambda^\dagger \), called the transpose. In our example, flipping gives

![Flipped Young diagram](image)

So \( (5, 3, 2)^\dagger = (3, 3, 2, 1, 1) \). In other words, the role of columns and rows has been interchanged. This is an important involution of \( \text{Par}(n) \) which we will use later.
1.1.2. \( \text{GL}_n \times \text{GL}_m \). We’d also like to consider representations of a product of general linear groups \( \text{GL}_n(\mathbb{C}) \times \text{GL}_m(\mathbb{C}) \). Much of the previous discussion applies. The Borel subgroup gets replaced with the product \( B \times B' \) where \( B \subset \text{GL}_n(\mathbb{C}) \) and \( B' \subset \text{GL}_m(\mathbb{C}) \) are the upper-triangular matrices in each group. Similarly, the maximal torus is replaced by the product \( T \times T' \). Weights are now pairs of vectors \((\mu_1, \ldots, \mu_n), (\mu'_1, \ldots, \mu'_m)\) and irreducible representations now correspond to highest weights which satisfy \( \mu_1 \geq \cdots \geq \mu_n \) and \( \mu'_1 \geq \cdots \geq \mu'_m \). The corresponding irreducible is denoted \( S_{\mu}(\mathbb{C}^n) \otimes S_{\mu'}(\mathbb{C}^m) \).

1.2. Multiplicity-free actions. This section is based on [Ho].

From the last part, we see that any representation \( W \) of \( \text{GL}_n(\mathbb{C}) \) is isomorphic to \( \bigoplus_{\lambda} S_{\lambda}(\mathbb{C}^n)^{\oplus m_{\lambda}} \) for some \( m_\lambda \geq 0 \). The integer \( m_\lambda \) is the multiplicity of \( S_{\lambda}(\mathbb{C}^n) \) in \( W \). We say that \( W \) is multiplicity-free if \( m_\lambda \leq 1 \) for all \( \lambda \). Similarly, any representation \( W \) of \( \text{GL}_n(\mathbb{C}) \times \text{GL}_m(\mathbb{C}) \) is isomorphic to \( \bigoplus_{\lambda, \lambda'} (S_{\lambda}(\mathbb{C}^n) \otimes S_{\lambda'}(\mathbb{C}^m))^{\oplus m_{\lambda, \lambda'}} \) and it is multiplicity-free if \( m_{\lambda, \lambda'} \leq 1 \) for all pairs \( \lambda, \lambda' \).

Given a vector space \( U \), we let \( \text{Sym}^d U^* \) denote the symmetric power of the dual space \( U^* \). This is the quotient of \( (U^*)^\otimes d \) by the subspace spanned by expressions \( u_1 \otimes \cdots \otimes u_d - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \) over all choices of permutations \( \sigma \) and \( u_1, \ldots, u_d \in U^* \). If we pick a basis \( v_1, \ldots, v_n \) for \( U \), then we get a dual basis \( x_1, \ldots, x_n \) for \( U^* \) and then \( \text{Sym}^d U^* \) is the space of degree \( d \) homogeneous polynomials in \( x_1, \ldots, x_n \) and has a basis given by all degree \( d \) monomials in the \( x_i \). If \( U \) is a representation of \( \text{GL}(V) \), then so is \( \text{Sym}^d U^* \). We define \( \text{Sym} U^* = \bigoplus_{d \geq 0} \text{Sym}^d U^* \), which is the space of all polynomials in \( x_1, \ldots, x_n \). These are functions on \( U \), so that given \( f \in \text{Sym} U^* \) and \( u \in U \), the evaluation \( f(u) \) makes sense. This is an infinite-dimensional representation of \( \text{GL}(V) \), but it is a direct sum of finite-dimensional ones, so everything we have said so far still applies.

**Example 1.2.1.** Consider the case \( U = V^* \) so that \( U^* = V \). I claim that \( \text{Sym}^d V \) is irreducible for any \( d \geq 0 \). Pick a basis \( x_1, \ldots, x_n \) for \( V \) and let \( T, B \) be the subgroups of diagonal and upper-triangular matrices with respect to this basis. Then the weight vectors are the monomials in \( x_1, \ldots, x_n \) and \( x_1^d \) is the only one which is a highest weight vector. So \( \text{Sym}^d V \) is irreducible with highest weight \( (d, 0, \ldots, 0) \). We see that \( \text{Sym} V \) is multiplicity-free. \( \square \)

We’d like to give a general criteria for \( \text{Sym} U^* \) to be multiplicity-free for a finite-dimensional representation \( U \). This requires some discussion of the Zariski topology.

1.2.1. Zariski topology. Let \( U \) be a vector space. A function \( f: U \to \mathbb{C} \) is polynomial if it can be written as a polynomial with respect to some (and hence, any) basis for \( U \). Given a (possibly infinite) set of polynomials \( I \), we define \( Z(I) \subset U \) to be the common solution set, i.e., \( Z(I) = \{ u \in U \mid f(u) = 0 \text{ for all } f \in I \} \). Note that if \( f, g \in I \) and \( h \) is any polynomial, then \( Z(I) = Z(I \cup \{ f + g \}) = Z(I \cup \{ fh \}) \). For that reason, we will always assume that \( I \) is closed under addition and multiplication by arbitrary polynomials, i.e., that \( I \) is an ideal.

Given a set of polynomials \( \{ f_i \} \), we let \( \langle f_i \rangle \) denote the smallest ideal containing the \( f_i \), i.e., \( \langle f_i \rangle = \{ \sum_j h_j f_j \} \) for all finite sums. We say that \( I \) is generated by \( \{ f_i \} \) if \( I = \langle f_i \rangle \).

An important fundamental fact:

**Theorem 1.2.2** (Hilbert basis theorem). Every ideal can be generated by a finite set of polynomials.
The sum of two ideals is \( I_1 + I_2 = \{ f + g \mid f \in I_1, \ g \in I_2 \} \) and the product \( I_1I_2 \) is the ideal generated by \( \{ fg \mid f \in I_1, \ g \in I_2 \} \). Infinite sums of ideals make sense: it is the ideal generated by all finite sums of elements in the ideals, but infinite products generally do not make sense.

The Zariski topology on \( U \) is the topology whose closed sets are the subsets of the form \( Z(I) \) for an ideal \( I \). It’s easy to check that this is a topology:

- Empty set is closed: \( \emptyset = Z(\{1\}) \)
- \( U \) is closed: \( U = Z(0) \)
- The intersection of closed sets is closed: \( \bigcap I_j Z(I_j) = Z(\sum_j I_j) \)
- A finite union of closed sets is closed: \( Z(I_1) \cup \cdots \cup Z(I_r) = Z(I_1 \cdots I_r) \)

Closed subsets of \( U \) (with the subspace topology) are called affine varieties. The coordinate ring of an affine variety \( X \subset U \) is denoted \( \mathbb{C}[X] \) and is the quotient of Sym \( U \) by the ideal of all polynomials which are identically 0 on \( X \). Note that \( \mathbb{C}[U] = \text{Sym} U^* \).

Open subsets of affine varieties (with the subspace topology) are called quasi-affine varieties. Every affine variety is quasi-affine but the converse does not generally hold.

**Example 1.2.3.** Note that \( \text{GL}_n(\mathbb{C}) \) can be interpreted as a quasi-affine variety: it is a subset of \( \mathbb{C}^{n^2} \) in the obvious way and its complement is the set of points whose determinant is 0. Since the determinant is a polynomial function in \( n^2 \) variables, we see that \( \text{GL}_n(\mathbb{C}) \) is an open subset.

Actually, we can realize it as an affine variety if we add 1 more variable. Consider \( \mathbb{C}^{n^2+1} \) and let \( t \) be the extra coordinate. Then

\[
Z(\langle t \det -1 \rangle) = \{ (g, \lambda) \mid g \in \text{GL}_n(\mathbb{C}), \ \lambda = 1/\det(g) \}.
\]

The extra information \( \lambda \) is not really extra, so the projection to the first \( n^2 \) coordinates identifies \( Z(\langle t \det -1 \rangle) \) with \( \text{GL}_n(\mathbb{C}) \). This is really an isomorphism of varieties, but we will not go into the details of what that means.

Just for reference, this means that \( \text{GL}_n(\mathbb{C}) \) is an algebraic group: an affine variety which has a group structure (such that the group product and inverse can be expressed by rational functions).

A topological space \( X \) is irreducible if, whenever \( X = X_1 \cup X_2 \) with both \( X_1, X_2 \) closed subsets, we must have \( X_1 = X \) or \( X_2 = X \).

**Proposition 1.2.4.** A vector space \( U \) with the Zariski topology is an irreducible topological space.

**Proof.** Suppose that \( U = Z(I_1) \cup Z(I_2) = Z(I_1I_2) \) for ideals \( I_1, I_2 \). Then \( I_1I_2 \) must be the 0 ideal: if there is a nonzero polynomial \( f \) in the product, then there is some point \( u \in U \) such that \( f(u) \neq 0 \). But this means that all pairwise products \( fg \) are 0 where \( f \in I_1 \) and \( g \in I_2 \). Since the product of nonzero polynomials is nonzero, this means that either \( I_1 = 0 \) or \( I_2 = 0 \), i.e., that \( Z(I_1) = U \) or \( Z(I_2) = U \).

**Proposition 1.2.5.** Every non-empty open subset \( Y \) of an irreducible space \( X \) is dense, i.e., if \( X' \supset Y \) is a closed subset of \( X \), then \( X = X' \).

**Proof.** With notation as in the statement, we have \( X = X' \cup (X \setminus Y) \). Since \( X \) is irreducible, either \( X' = X \) or \( X \setminus Y = X \). The latter means that \( Y \) is empty, so it must be that \( X' = X \).
Now we come back to multiplicity-free spaces. Given an affine variety \( X \) in a representation \( U \) of \( \GL(V) \) and a subgroup \( G \subset \GL(V) \), we define the \( G \)-orbits of \( X \) to be the equivalence classes of the equivalence relation on \( X \) given by \( u \sim u' \) if \( u' = g \cdot u \) for some \( g \in G \).

**Theorem 1.2.6.** Let \( X \) be an affine variety in some representation of \( \GL(V) \) and let \( B \) be a Borel subgroup. Suppose that there is a \( B \)-orbit \( Y \) on \( X \) which is dense. Then

(a) \( C[X] \) is a multiplicity-free representation.

(b) Let \( \lambda \) be the weight of a highest weight vector of \( C[X] \). Pick \( u \in Y \) and let \( H \) be the stabilizer of \( u \), i.e., \( H = \{ g \in \GL(V) \mid h \cdot u = u \} \). Then \( \lambda(h) = 1 \) for all \( h \in H \cap B \).

Proof. (a) Pick two highest weight vectors \( f, g \in C[X] \) with the same weight \( \lambda \). Pick a point \( u \in Y \). Note that \( f(b \cdot u) = (b^{-1} \cdot f)(u) = \lambda(b^{-1})f(u) \). If \( f(u) = 0 \), then \( f \) is 0 on all of \( Y \), which means it is the zero polynomial since \( Z(f) \supset Y \). Similarly, \( g(u) \neq 0 \), so there is a nonzero scalar \( \alpha \) so that \( g(u) = \alpha f(u) \). But then \( g - \alpha f \) is a highest weight vector of weight \( \lambda \), but \( (g - \alpha f)(u) = 0 \), so the previous reasoning shows that \( g - \alpha f \) is the zero polynomial, i.e., that \( f, g \) must be scalar multiples of each other.

(b) Continuing the same notation, if \( h \in H \cap B \), then \( f(u) = f(h \cdot u) = \lambda(h^{-1})f(u) \). Since \( f(u) \neq 0 \), this means \( \lambda(h^{-1}) = 1 \). Since \( H \cap B \) is a group and \( h \) was arbitrary, this implies \( \lambda(h) = 1 \) for all \( h \in H \cap B \).

Finally, here’s a useful observation.

**Proposition 1.2.7.** Let \( X \) be an irreducible affine variety in some representation of \( \GL(V) \). If \( \lambda \) and \( \mu \) are highest weights for irreducible representations appearing in \( C[X] \), then so is \( \lambda + \mu \).

Proof. Let \( f \) and \( g \) be highest weight vectors of weights \( \lambda \) and \( \mu \) in \( C[X] \). We claim that \( fg \neq 0 \). If so, then \( Z(f) \cup Z(g) = X \) which means that \( X = Z(f) \) or \( X = Z(g) \) since \( X \) is irreducible. But both \( f \) and \( g \) are nonzero so this is impossible. Next, for any \( b \in B \), we have \( b \cdot (fg) = (b \cdot f)(b \cdot g) = \lambda(b)\mu(b)fg = (\lambda + \mu)(b)fg \).

These results apply just as well if we have a product of general linear groups with the product of Borel subgroups replacing \( B \).

1.2.2. Example: generic matrices. Pick integers \( m \geq n \). We consider the product \( \GL_n(\mathbb{C}) \otimes \GL_m(\mathbb{C}) \) with the representation \( U = (\mathbb{C}^n \otimes \mathbb{C}^m)^* \) being the space of \( n \times m \) matrices. The action is given by

\[
(g, h) \cdot u = (g^{-1})^Tuh^{-1}.
\]

Let \( J \) be the \( n \times m \) matrix with \( J_{i,i} = 1 \) for \( i = 1, \ldots, n \) and 0’s elsewhere. Let \( B \subset \GL_n(\mathbb{C}) \) be the subgroup of upper-triangular matrices, and similarly, let \( B' \subset \GL_m(\mathbb{C}) \) be the subgroup of upper-triangular matrices.

Let \( A_i \) be the upper-left \( i \times i \) submatrix of the generic matrix \( \varphi = (\varphi_{ij}) \) and let \( f_i = \det A_i \).

**Proposition 1.2.8.** \( f_i \) is a highest weight vector with weight \((1, \ldots, 1, 0, \ldots, 0), (1, \ldots, 1, 0, \ldots, 0) \) (the number of 1’s in each vector is \( i \)).

Proof. Pick upper-triangular matrices \( g \in \GL_n(\mathbb{C}) \) and \( h \in \GL_m(\mathbb{C}) \) and write \( g = \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix} \) and \( h = \begin{bmatrix} x_2 & y_2 \\ 0 & z_2 \end{bmatrix} \) where \( x_1, x_2 \) are \( i \times i \). Then \( (g, h) \cdot f_i \) is the determinant of the upper-left \( i \times i \) submatrix of \( g^T \varphi h \), which is \( \det(x_1^T A_i x_2) = \det(x_1) \det(x_2)f_i \). In particular, it is a
highest weight vector. If $g, h$ are diagonal, then $\det(x_1) \det(x_2)$ is just the product of the first $i$ entries of each of $g$ and $h$, so we get the weight also. \qed

Lemma 1.2.9. The $B \times B'$ orbit containing $J$ is open and dense.

Proof. We claim that the orbit is precisely the set of matrices $A$ such that $f_i(A) \neq 0$ for $i = 1, \ldots, n$. It is easy to see that being in the orbit implies the condition on submatrices, so we just prove the reverse direction. Before proving the claim, note that this shows that the orbit is $\bigcap_{i=1}^n (U \setminus Z(f_i))$, which is open (denseness follows from Propositions 1.2.4 and 1.2.5).

We first handle the case $n = m$. Then $J$ is just the identity matrix, so the $B \times B'$ orbit of $J$ is the set of matrices $A$ with an LU factorization, i.e., $A = LU$ where $L$ is lower-triangular and invertible with 1’s on the diagonal, and $U$ is upper-triangular and invertible.

We proceed by induction on $n$. The case $n = 1$ is immediate, so for the general case, write $A = \begin{bmatrix} A' & b \\ c & d \end{bmatrix}$ where $A'$ is $(n-1) \times (n-1)$. Then all of the upper left submatrices of $A'$ are invertible and so we have a factorization $A' = LU'$. Then we have

$$
\begin{bmatrix} A' & b \\ c & d \end{bmatrix} = \begin{bmatrix} L' & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} U' & y \\ 0 & z \end{bmatrix}
$$

where $y = (L')^{-1}b$, $x = c(U')^{-1}$, and $z = d - xy$. We let $L$ and $U$ be these last two matrices. Then $L$ is lower-triangular and invertible with 1’s on the diagonal and $U$ is upper-triangular and invertible (since $U = L^{-1}A$).

For the general case $m \geq n$, write $A = [A', A'']$ where $A'$ is $n \times n$ and its upper left submatrices are invertible. Then we can factor it as $A' = LU$ as above. Then we have

$$
A = LJ \begin{bmatrix} U & L^{-1}A'' \\ 0 & \text{id}_{m-n} \end{bmatrix}.
$$

Lemma 1.2.10. If $(\lambda, \lambda')$ is the weight of a highest weight vector in $\text{Sym} U^*$, then $\lambda_i = \lambda'_i$ for $1 \leq i \leq n$ and $\lambda'_j = 0$ for $j > n$. Also, $\lambda_n \geq 0$.

Proof. The stabilizer of $J$ contains pairs of diagonal matrices $(g, h)$ where the entries of $h$ are $x_1, \ldots, x_n$ and the entries of $g$ are $x_1^{-1}, \ldots, x_n^{-1}$. By Theorem 1.2.6, we see that $x_1^{\lambda_1-\lambda_i} \cdots x_n^{\lambda_n-\lambda_n} x_{n+1}^{\lambda_{n+1}} \cdots x_m^{\lambda_m} = 1$ for all $x_1, \ldots, x_m$. This forces all of the exponents to be 0.

Finally, all weights attached to weight vectors in $\text{Sym} U^*$ are non-negative. This follows from the formula for the action on $U$. \qed

Finally, each pair $(\lambda, \lambda')$ as above is the highest weight for some irreducible representation in $\text{Sym} U^*$ by Proposition 1.2.7 since every partition $\lambda$ is a sum of vectors of the form $(1, 1, \ldots, 1, 0, \ldots, 0)$.

Corollary 1.2.11 (Cauchy identity). We have an isomorphism of $\text{GL}_n(\mathbb{C}) \times \text{GL}_m(\mathbb{C})$ representations

$$
\text{Sym}(\mathbb{C}^n \otimes \mathbb{C}^m) \cong \bigoplus_{\lambda} S^\lambda(\mathbb{C}^n) \otimes S^\lambda(\mathbb{C}^m)
$$

where the sum is over all integer partitions $\lambda_1 \geq \cdots \lambda_n \geq 0$ and the second $\lambda$ is understood to have $m - n$ 0’s added at the end.
1.2.3. Example: symmetric matrices. The space $U$ of $n \times n$ symmetric matrices has an action of $\text{GL}_n(\mathbb{C})$ via

$$g \cdot X = (g^{-1})^T X g^{-1}.$$ 

Let $f_1$ be the function which takes the $(1,1)$-entry of a symmetric matrix. If $g$ is upper-triangular, then $(g \cdot f_1)(X) = f_1(g^{-1} \cdot X) = g_{1,1}^2 f_1(X)$, so $f_1$ is a highest weight vector of $U^*$ with highest weight $(2,0,\ldots,0)$. So $U^*$ contains a copy of $\text{Sym}^2 \mathbb{C}^n$ as a $\text{GL}_n(\mathbb{C})$-representation. Since they have the same dimension, $U^* \cong \text{Sym}^2 \mathbb{C}^n$.

We want to repeat our analysis from the previous section to this new example.

Let $f_i$ be the determinant of the upper-left $i \times i$ submatrix of a symmetric matrix.

**Proposition 1.2.12.** $f_i$ is a highest weight vector with weight $(2,\ldots,2,0,\ldots,0)$ (the number of 2’s is $i$).

**Proof.** Let $g \in \text{GL}_n(\mathbb{C})$ be an upper triangular matrix and write it as $g = \begin{bmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & g_5 \\ 0 & 0 & g_6 \end{bmatrix}$ where $g_1$ is $i \times i$. Let $X$ be a symmetric matrix and write it as $X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ where $A$ is $i \times i$. Then $(g \cdot f_i)(X) = f_i(g^{-1} \cdot X) = f_i(g^T X g)$. Then the upper left $i \times i$ submatrix of $g^T X g$ is $g_1^T A g_1$, and its determinant is $\det(g_1)^2 f_i(X)$, which shows that $f_i$ is a highest weight vector of weight $(2,\ldots,2,0,\ldots,0)$.

**Lemma 1.2.13.** The $B$-orbit of the identity matrix $I$ is open and dense.

**Proof.** The $B$-orbit of $I$ is the set of symmetric matrices which can be factored as $g^T g$ for some invertible upper triangular matrix $g$. We claim that this is the set of symmetric matrices $X$ such that $f_i(X) \neq 0$ for $i = 1,\ldots,n$, and prove it by induction on $n$.

If $n = 1$, this is clear. Otherwise, write $X = \begin{bmatrix} X' & y \\ y^T & z \end{bmatrix}$ where $X'$ has size $(n-1) \times (n-1)$. By induction, $f_i(X') \neq 0$ for $i = 1,\ldots,n-1$ and so we have $X' = h^T h$ for an invertible upper triangular $(n-1) \times (n-1)$ matrix $h$. Then we have

$$\begin{bmatrix} X' & y \\ y^T & z \end{bmatrix} = \begin{bmatrix} h^T & 0 \\ y^T h^{-1} & \alpha \end{bmatrix} \begin{bmatrix} h & (h^T)^{-1} y \\ 0 & \alpha \end{bmatrix}$$

where $\alpha^2 = z - y^T h^{-1} (h^T)^{-1} y$. The new matrices we produced are automatically invertible since their product is invertible (by the assumption $f_n(X) \neq 0$).

This implies that the orbit is Zariski open (and hence dense).

**Lemma 1.2.14.** If $\lambda$ is a weight of a highest weight vector in $\text{Sym} U^*$, then $\lambda_i$ is even for all $i$ and $\lambda_n \geq 0$.

**Proof.** We use Theorem 1.2.6. Let $h$ be the diagonal matrix with 1’s on the diagonal except for a $-1$ in position $i$. Then $h^T h = I$, and $\lambda(h) = (-1)^{\lambda_i}$, so $\lambda_i$ is even. Since all weights of $\text{Sym}(U^*)$ are non-negative, we also get that $\lambda_n \geq 0$.

**Corollary 1.2.15.** We have an isomorphism of $\text{GL}_n(\mathbb{C})$ representations

$$\text{Sym}(\text{Sym}^2 \mathbb{C}^n) \cong \bigoplus_{\lambda} \mathbb{S}_{2\lambda}(\mathbb{C}^n)$$

where the sum is over all integer partitions $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. 

Proof. We have shown that $\text{Sym}(\text{Sym}^2 \mathbb{C}^n)$ is multiplicity-free and all highest weights are of the form $2\lambda$ for an integer partition $\lambda$. It remains to show they all appear in the decomposition, but we can get all of them by multiplying the functions $f_1, \ldots, f_n$ with various multiplicities. 

1.2.4. Example: skew-symmetric matrices. Given a vector space $V$ and a positive integer $d$, the $d$th exterior power $\wedge^d V$ is the quotient of $V^\otimes d$ by the subspace spanned by elements of the form $v_1 \otimes \cdots \otimes v_d$ where $v_i = v_j$ for some $i \neq j$. Note that the implies that swapping two elements introduces a sign in $\wedge^d V$, for example when $d = 2$ we have:

$$0 = (v_1 + v_2) \otimes (v_1 + v_2) = v_1 \otimes v_1 + v_1 \otimes v_2 + v_2 \otimes v_1 + v_2 \otimes v_2 = v_1 \otimes v_2 + v_2 \otimes v_1$$

and the general case is similar (but with more cumbersome notation). The coset of $v_1 \otimes \cdots \otimes v_d$ is denoted $v_1 \wedge \cdots \wedge v_d$; from what we said it satisfies $v_1 \wedge \cdots \wedge v_d = (\text{sgn} \, \sigma)v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(d)}$ for any permutation $\sigma$. If $e_1, \ldots, e_r$ is a basis for $V$, then a basis for $\wedge^d V$ is given by $\{e_{i_1} \wedge \cdots \wedge e_{i_d} \mid 1 \leq i_1 < \cdots < i_d \leq r\}$ so that it has dimension $\binom{r}{d}$.

The space $U$ of $n \times n$ skew-symmetric matrices has an action of $\text{GL}_n(\mathbb{C})$ via

$$g \cdot X = (g^{-1})^T X g^{-1}.$$ 

Let $f_1$ be the function which takes the $(1,2)$-entry of a skew-symmetric matrix. If $g$ is upper-triangular, then $(g \cdot f_1)(X) = f_1(g^{-1} \cdot X) = g_{1,1}g_{2,2}f_1(X)$, so $f_1$ is a highest weight vector of $U^*$ with highest weight $(1,0,0,\ldots,0)$. So $U^*$ contains a copy of $\wedge^2 \mathbb{C}^n$ as a $\text{GL}_n(\mathbb{C})$-representation. Since they have the same dimension, $U^* \cong \wedge^2 \mathbb{C}^n$.

We want to repeat our analysis from the previous section to this new example. Before doing this, we need a short digression on Pfaffians.

First, given a skew-symmetric $n \times n$ matrix $X$, we have $\det(X) = \det(X^T) = \det(-X) = (-1)^n \det(X)$, so $\det X = 0$ if $n$ is odd. In particular, the rank of a skew-symmetric matrix is always even. Let $X$ have size $2i \times 2i$. Let $\Pi$ be the set of permutations $\sigma$ of $2i$ that satisfy $\sigma(1) < \sigma(3) < \cdots < \sigma(2i - 1)$ and $\sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \ldots, \sigma(2i - 1) < \sigma(2i)$. The Pfaffian of $X$ is defined by

$$\text{Pf}(X) = \sum_{\sigma \in \Pi} (\text{sgn} \, \sigma) X_{\sigma(1),\sigma(2)}X_{\sigma(3),\sigma(4)} \cdots X_{\sigma(2i-1),\sigma(2i)}.$$ 

The formula is not so important, but here are two important properties (whose proofs we will omit):

- $(\text{Pf} \, X)^2 = \det X$
- $\text{Pf}(gXg^T) = (\det g)(\text{Pf} \, X)$ for any $g \in \text{GL}_{2i}(\mathbb{C})$.

Let $f_i$ be the Pfaffian of the upper-left $2i \times 2i$ submatrix of a skew-symmetric matrix. It is a polynomial function of degree $i$.

**Proposition 1.2.16.** $f_i$ is a highest weight vector with weight $(1,\ldots,1,0,\ldots,0)$ (the number of $1$’s is $2i$).

**Proof.** Let $g \in \text{GL}_n(\mathbb{C})$ be an upper triangular matrix and write it as $g = \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix}$ where $g_1$ is $i \times i$. Let $X$ be a skew-symmetric matrix and write it as $X = \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix}$ where $A$ is $2i \times 2i$. Then $(g \cdot f_i)(X) = f_i(g^{-1} \cdot X) = f_i(g^TXg)$. The upper left $2i \times 2i$ submatrix of $g^TXg$
is $g_1^T Ag_1$, and its Pfaffian is $\det(g_1)f_i(X)$, which shows that $f_i$ is a highest weight vector of weight $(1, \ldots, 1, 0, \ldots, 0)$.

Consider the $2 \times 2$ matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. If $n$ is even, let $J_n$ be the block diagonal matrix consisting of $n/2$ copies of this matrix. If $n$ is odd, then let $J_n$ be the block diagonal matrix consisting of $(n - 1)/2$ copies of this matrix and one extra 0 at the end.

**Lemma 1.2.17.** The $B$-orbit of the identity matrix $J_n$ is open and dense.

**Proof.** The $B$-orbit of $J_n$ is the set of skew-symmetric matrices which can be factored as $g^T J_n g$ for some invertible upper triangular matrix $g$. We claim that this is the set of skew-symmetric matrices $X$ such that $f_i(X) \neq 0$ for $i = 1, \ldots, [n/2]$, and prove it by induction on $n$.

We first handle the case when $n$ is even. If $n = 2$, this is clear. Otherwise, write $X = \begin{bmatrix} X' & y \\ -y^T & z \end{bmatrix}$ where $X'$ has size $(n-2) \times (n-2)$. By induction, $f_i(X') \neq 0$ for $i = 1, \ldots, (n-2)/2$ and so we have $X' = h^T J_{n-2} h$ for an invertible upper triangular $(n - 2) \times (n - 2)$ matrix $h$. Then we have

$$\begin{bmatrix} X' & y \\ -y^T & z \end{bmatrix} = \begin{bmatrix} h^T & 0 \\ \alpha^T & \beta^T \end{bmatrix} \begin{bmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} h & \alpha \\ 0 & \beta \end{bmatrix}$$

where $\alpha = (h^T J_{n-2})^{-1} y$ and $\beta = \begin{bmatrix} z_{1,2} & 0 \\ 0 & 1 \end{bmatrix}$. The new matrices we produced are automatically invertible since their product is invertible (by the assumption $f_{n/2}(X) \neq 0$).

Now we do the case when $n$ is odd. Write $X = \begin{bmatrix} X' & y \\ -y^T & 0 \end{bmatrix}$ where $X'$ has size $(n-1) \times (n-1)$. Since $n - 1$ is even, we can factor $X' = h^T J_{n-1} h$ for an upper triangular invertible matrix $h$. Then we have

$$\begin{bmatrix} X' & y \\ -y^T & 0 \end{bmatrix} = \begin{bmatrix} h^T & 0 \\ \alpha^T & 1 \end{bmatrix} \begin{bmatrix} J_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h & \alpha \\ 0 & 1 \end{bmatrix}$$

where $\alpha = (h^T J_{n-1})^{-1} y$ and clearly $\begin{bmatrix} h & \alpha \\ 0 & 1 \end{bmatrix}$ is invertible and upper-triangular.

This implies that the orbit is Zariski open (and hence dense).  

**Lemma 1.2.18.** If $\lambda$ is a weight of a highest weight vector in $\text{Sym} U^*$, then $\lambda_i^1$ is even for all $i$ and $\lambda_n \geq 0$.

**Proof.** We use Theorem 1.2.6. If $n = 2m$ is even, let $h$ be the diagonal matrix with entries $x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}$. Then $h^T J_n h = J_n$ and $\lambda(h) = x_1^{\lambda_1-\lambda_2}x_2^{\lambda_3-\lambda_4} \cdots x_m^{\lambda_{m-1}-\lambda_n}$. If this is 1 for all choices of $x_1, \ldots, x_m$, then we must have $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4$, etc. This is the same as saying that $\lambda_i^1$ is even for all $i$.

If $n = 2m + 1$ is odd, let $h$ be the diagonal matrix with entries $x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}, x_{m+1}$. Then $h^T J_n h = J_n$ and $\lambda(h) = x_1^{\lambda_1-\lambda_2}x_2^{\lambda_3-\lambda_4} \cdots x_m^{\lambda_{m-2}-\lambda_{m-1}}x_{m+1}^{\lambda_n}$. If this is 1 for all choices of $x_1, \ldots, x_{m+1}$, then we must have $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4$, etc. and $\lambda_n = 0$. This is the same as saying that $\lambda_i^1$ is even for all $i$.

In either case, all weights of $\text{Sym}(U^*)$ are non-negative, so we also get that $\lambda_n \geq 0$. 

\[ \square \]
Corollary 1.2.19. We have an isomorphism of $\text{GL}_n(\mathbb{C})$ representations

$$\text{Sym}(\bigwedge^2 \mathbb{C}^n) \cong \bigoplus_{\lambda} S^{(2\lambda)}_\lambda(\mathbb{C}^n)$$

where the sum is over all integer partitions such that $2\lambda_1 \leq n$.

Proof. We have shown that $\text{Sym}(\bigwedge^2 \mathbb{C}^n)$ is multiplicity-free and all highest weights are of the form $(2\lambda)^\dagger$ for an integer partition $\lambda$. It remains to show they all appear in the decomposition, but we can get all of them by multiplying the functions $f_1, \ldots, f_{\lceil n/2 \rceil}$ with various multiplicities.

\[\square\]

1.3. Schur functors. This section will mostly give facts without proofs.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of $d$ and set $\mu = \lambda^\dagger = (\mu_1, \ldots, \mu_s)$. The Schur functor $S_\lambda V$ is defined to be the image of the following composition:

$$\bigwedge^{\mu_1} V \otimes \cdots \otimes \bigwedge^{\mu_s} V \rightarrow V^{\otimes \mu_1} \otimes \cdots \otimes V^{\otimes \mu_s}$$

$$\rightarrow V^{\otimes \lambda_1} \otimes \cdots \otimes V^{\otimes \lambda_r}$$

$$\rightarrow \text{Sym}^{\lambda_1} V \otimes \cdots \otimes \text{Sym}^{\lambda_r} V,$$

where the first map is given by comultiplication:

$$\bigwedge^d V \rightarrow V^{\otimes d}$$

$$(v_1 \wedge \cdots \wedge v_d) \mapsto \sum_{\sigma \in \mathfrak{S}_d} (\text{sgn} \sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$$

($\mathfrak{S}_d$ denotes the symmetric group on $d$ letters), the second map is a certain reordering that we will explain shortly, and the third map is multiplication, i.e., the quotient map $V^{\otimes d} \rightarrow \text{Sym}^d V$.

The reordering is best understood in terms of Young diagrams, which we will illustrate with an example.

Example 1.3.1. Consider $\lambda = (3,2)$ so that $\mu = (2,2,1)$. Then $\bigwedge^2 V \otimes \bigwedge^2 V \otimes V$ is spanned by elements of the form $(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \otimes v_5$. We can record element of $V^{\otimes 5}$ by putting vectors into the boxes of $Y(\lambda)$. The order we do this in depends: if we write $V^{\otimes 5} = V^{\otimes \mu_1} \otimes \cdots \otimes V^{\otimes \mu_s}$, then we will think of these as the columns of $Y(\lambda)$. On the other hand, writing $V^{\otimes 5} = V^{\otimes \lambda_1} \otimes \cdots \otimes V^{\otimes \lambda_r}$, we will instead think of these as the rows of the $Y(\lambda)$. The map then looks as follows:

$$(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \otimes v_5 \mapsto \begin{bmatrix} v_1 v_3 v_5 & v_2 v_3 v_5 & v_1 v_4 v_5 & v_2 v_4 v_5 \\ v_2 v_4 & v_1 v_4 & v_2 v_3 & v_1 v_3 \end{bmatrix}$$

$$\mapsto v_1 v_3 v_5 \otimes v_2 v_4 - v_2 v_3 v_5 \otimes v_1 v_4 - v_1 v_4 v_5 \otimes v_2 v_3 + v_2 v_4 v_5 \otimes v_1 v_3. \square$$

Since all of the maps are $\text{GL}(V)$-equivariant, we see that $S_\lambda(V)$ is a $\text{GL}(V)$-representation. It follows immediately from the definition that $S_\lambda(V) = 0$ if $\ell(\lambda) > \text{dim} V$ since the corresponding exterior power $\bigwedge^{\mu_1} V$ is 0.

Example 1.3.2. There are two extreme cases that we already know. If $\lambda = (d)$, then the map becomes the quotient map $V^{\otimes d} \rightarrow \text{Sym}^d V$ so that $S_{(d)} V = \text{Sym}^d V$. On the other
hand, if \( \lambda = (1^d) \), then the map becomes the comultiplication map \( \wedge^d V \to V \otimes V \), which is injective, so \( S_{(1^d)} V = \wedge^d V \).

Fix a basis \( e_1, \ldots, e_n \) for \( V \). We would like to find a basis for \( S_\lambda V \). Given a tableau \( T \) on \( Y(\lambda) \), i.e., a filling of the boxes of \( Y(\lambda) \) with the numbers \( 1, \ldots, n \), we get a vector in \( \wedge^{\mu_1} V \otimes \cdots \otimes \wedge^{\mu_s} V \) by taking

\[
(e_{T_{1,1}} \wedge e_{T_{2,1}} \wedge \cdots \wedge e_{T_{\mu_1,1}}) \otimes \cdots \otimes (e_{T_{1,s}} \wedge \cdots \wedge e_{T_{\mu_s,s}});
\]

let \( e_T \) be its image in \( S_\lambda V \).

We say that \( T \) is semistandard if \( T_{i,j} \leq T_{i,j+1} \) and \( T_{i,j} < T_{i+1,j} \) for all \( i, j \) where that makes sense.

**Theorem 1.3.3.** \( \{e_T \mid T \text{ is semistandard}\} \) is a basis for \( S_\lambda V \).

The proof is elementary, but complicated, so we will omit it.

The \( e_T \) are all weight vectors of weight \( \mu(T) \) where \( \mu(T)_i \) is the number of times that \( i \) appears in the tableau \( T \). Note that different tableau can have the same weight. Consider the tableau \( T \) where the boxes in row \( i \) are filled with \( i \). Then this is a highest weight vector of weight \( \lambda \) since it is the image of a tensor product of highest weight vectors in the exterior powers. In fact, there are no other highest weight vectors (we omit the proof), so we conclude the following theorem:

**Theorem 1.3.4.** \( S_\lambda V \) is an irreducible polynomial representation of \( GL(V) \) of highest weight \( \lambda \).

We see that the dimension of \( S_\lambda V \) is the number of semistandard Young tableau (SSYT) of shape \( \lambda \). We give two formulas for this quantity:

**Theorem 1.3.5.**

\[
\dim S_\lambda(C^n) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\]

Given a box \((i, j)\) in the Young diagram of \( \lambda \), its hook is the set of boxes to the right and below it (including itself). Its hook length \( h(i, j) \) is the number of boxes in the hook. Below, we list the hook lengths for the partition \((6, 3, 1)\):

\[
\begin{array}{ccccccc}
8 & 6 & 5 & 3 & 2 & 1 \\
4 & 2 & 1 \\
1
\end{array}
\]

Given a box \((i, j) \in Y(\lambda)\), define its content to be \( c(i, j) = j - i \).

**Theorem 1.3.6** (Hook-content formula).

\[
\dim S_\lambda(C^n) = \prod_{(i,j)\in Y(\lambda)} \frac{n + c(i,j)}{h(i,j)}.
\]

[⋆ Steven: insert example ⋆]
1.4. Symmetric polynomials and functions.

**Lemma 1.4.1.** \( \text{char}(\rho)(x_1, \ldots, x_n) \) is symmetric, i.e., for any permutation \( \sigma \), we have \( \text{char}(\rho)(x_1, \ldots, x_n) = \text{char}(\rho)(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \).

**Proof.** Each permutation \( \sigma \) corresponds to a permutation matrix \( M(\sigma) \): this is the matrix with a 1 in row \( \sigma(i) \) and column \( i \) for \( i = 1, \ldots, n \) and 0’s everywhere else. Then

\[
M(\sigma)^{-1}\text{diag}(x_1, \ldots, x_n)M(\sigma) = \text{diag}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
\]

Now use that the trace of a matrix is invariant under conjugation:

\[
\text{char}(\rho)(x_1, \ldots, x_n) = \text{Tr}(\rho(\text{diag}(x_1, \ldots, x_n)))
= \text{Tr}(\rho(M(\sigma)^{-1}\text{diag}(x_1, \ldots, x_n)\rho(M(\sigma)))
= \text{Tr}(\rho(\text{diag}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})))
= \text{char}(\rho)(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \quad \square
\]

By Proposition 1.1.6, we know that every representation becomes polynomial after tensoring with a large enough power of the determinant. This means that every character is of the form \( f(x_1, \ldots, x_n)/(x_1 \cdots x_n)^d \) where \( f \) is a symmetric polynomial (i.e., invariant under permutations of the variables).

We denote \( \Lambda(n) \) to be the set of symmetric polynomials in \( n \) variables \( x_1, \ldots, x_n \) with integer coefficients. This is a ring under usual addition and multiplication and contains the characters of polynomial representations of \( \text{GL}_m(C) \).

We can do something analogous for representations of \( \text{GL}_n(C) \times \text{GL}_m(C) \): let \( \Lambda(n, m) \) be the set of polynomials in two sets of variables \( x_1, \ldots, x_n, y_1, \ldots, y_m \) with integer coefficients which are symmetric in each set separately.

1.4.1. **Schur polynomials.** The **Schur polynomial** \( s_\lambda(x_1, \ldots, x_n) \) is the character of \( S_\lambda(C^n) \). These form a basis for \( \Lambda(n) \) (can be proven combinatorially, see Math 202B, or using general results about representation theory, see Math 251AB). We can write it as a sum of \( x^\mu(T) \) over SSYT \( T \) of shape \( \lambda \) where \( \mu(T) \) is the weight of \( T \). The **Weyl character formula** can be translated to give a more compact formula which we explain now.

Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a non-negative integer sequence. Define

\[
a_\alpha = \det(x_1^{\alpha_j})_{i,j=1}^n = \det \begin{pmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_n} \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_2^{\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} & x_n^{\alpha_2} & \cdots & x_n^{\alpha_n} \end{pmatrix} = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma)\sigma(x^\alpha).
\]

Note that \( a_\alpha \) is skew-symmetric: if we permute \( a_\alpha \) by a permutation \( \sigma \in \mathfrak{S}_n \), then it changes by \( \text{sgn}(\sigma) \). Let \( \rho = (n-1, n-2, \ldots, 1, 0) \).

**Lemma 1.4.2.** (a) \( \prod_{1 \leq i < j \leq n} (x_i - x_j) \) divides every skew-symmetric polynomial in \( x_1, \ldots, x_n \).
(b) \( a_\rho = \prod_{1 \leq i < j \leq n} (x_i - x_j) \).

**Proof.** (a) Let \( f(x_1, \ldots, x_n) \) be skew-symmetric and let \( \sigma \) be the transposition \( (i, j) \). Then \( \sigma f = -f \). However, \( \sigma f \) and \( f \) are the same if we replace \( x_j \) by \( x_i \), so this says that specializing \( x_j \) to \( x_i \) gives 0, i.e., \( f \) is divisible by \( (x_i - x_j) \). This is true for any \( i, j \), so this proves (a).
(b) $a_\rho$ is divisible by $\prod_{1 \leq i < j \leq n}(x_i - x_j)$ since it is skew-symmetric. But also note that both are polynomials of degree $1 + 2 + \cdots + (n - 1) = \binom{n}{2}$, so they are equal up to some integer multiple. The coefficient of $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ for both is 1, so they are actually the same. □

Define $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

**Theorem 1.4.3** (Weyl character formula for $\text{GL}_n(\mathbb{C})$). Given a partition $\lambda$,

$$s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho} = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\prod_{1 \leq i < j \leq n}(x_i - x_j)}.$$

Remark 1.4.4. The Weyl character formula usually takes this form:

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(x^{\lambda+\rho}) \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(x^\rho).$$

**1.4.2. Cauchy identity and some plethysm.** From the way we defined it, it makes sense to apply a Schur functor to any representation $W$ of $\text{GL}_n(\mathbb{C})$: $S_\lambda(W)$. If we think of this as a representation of $\text{GL}_n(\mathbb{C})$ (rather than $\text{GL}(W)$), then we can get an expression for its character as follows. Let $w_1, \ldots, w_N$ be the weights (repeated as necessary) of $W$. Then

$$\text{char}(S_\lambda(W))(x_1, \ldots, x_n) = s_\lambda(x^{w_1}, \ldots, x^{w_N}).$$

We denote this by $s_\lambda \circ \text{char}(W)$. It is then possible to write this as a linear combination of Schur polynomials in the $x_i$ which tells us how the representation decomposes into irreducible representations. Explicitly doing this is hard in general, though in specific examples it can be done with basic linear algebra. More generally, any symmetric polynomial $f$ is a linear combination of Schur polynomials $\sum c_\lambda s_\lambda(x)$, so we can $f \circ \text{char}(W) = \sum \lambda c_\lambda s_\lambda \circ \text{char}(W)$ for any symmetric polynomial $f$. Actually, it is possible to extend the definition further and allow $\text{char}(W)$ to be any symmetric polynomial as well, though we won’t do that. This operation is called **plethysm**.

Now we’ll translate the multiplicity-free actions studied earlier into symmetric polynomial identities. To do that, we will work with power series in a new variable $t$ whose coefficients lie in $\Lambda(n)$. First we need a simple identity. Let $h_d(x) = \text{char}(\text{Sym}^d \mathbb{C}^n)$.

**Lemma 1.4.5.**

$$\sum_{d \geq 0} h_d(x)t^d = \frac{1}{\prod_{i=1}^n(1 - x_it)}.$$

**Proof.** The left side is the sum of $x_1^{d_1} \cdots x_n^{d_n}t^{d_1+\cdots+d_n}$ over all choices of non-negative integers $d_1, \ldots, d_n$. Using the geometric series, the right side is the product

$$\prod_{i=1}^n(\sum_{d_i \geq 0} x_i^{d_i}t^{d_i}),$$

which is the same when expanded out. □

**Theorem 1.4.6** (Cauchy identity).

$$\prod_{i=1}^n \prod_{j=1}^m (1 - x_iy_j)^{-1} = \sum_{\lambda} s_\lambda(x_1, \ldots, x_n)s_\lambda(y_1, \ldots, y_m)$$

where the sum is over all integer partitions with $\lambda_1 \geq \cdots \geq \lambda_{\min(n,m)} \geq 0$. 

Proof. By Corollary 1.2.11, we have
\[ \sum_{d \geq 0} h_d(x_1 y_1, \ldots, x_n y_m) t^d = \sum_{\lambda} s_{\lambda}(x_1, \ldots, x_n) s_{\lambda}(y_1, \ldots, y_m). \]

Use the previous lemma to replace the left side. \qed

Similarly, we get the following two identities using Corollary 1.2.15 and Corollary 1.2.19:

**Theorem 1.4.7.**
\[
\prod_{1 \leq i \leq j \leq n} (1 - x_i x_j t)^{-1} = \sum_{\lambda} s_{2\lambda}(x_1, \ldots, x_n),
\]
\[
\prod_{1 \leq i < j \leq n} (1 - x_i x_j t)^{-1} = \sum_{\lambda} s_{(2\lambda)^t}(x_1, \ldots, x_n).
\]

1.4.3. **Symmetric functions.** We have maps \( \pi_n : \Lambda(n+1) \to \Lambda(n) \) obtained by \( f(x_1, \ldots, x_n, x_{n+1}) \mapsto f(x_1, \ldots, x_n, 0) \). This gives us a way to compare representations of \( \text{GL}_{n+1}(\mathbb{C}) \) with \( \text{GL}_n(\mathbb{C}) \).

It follows from our interpretation in terms of SSYT that \( \pi_n(s_{\lambda}(x_1, \ldots, x_{n+1})) = s_{\lambda}(x_1, \ldots, x_n) \).

We define \( \Lambda \) to be the graded inverse limit of the system \( \{ \pi_n : \Lambda(n+1) \to \Lambda(n) \} \). Explicitly, a degree \( d \) element \( f \in \Lambda \) is a sequence of degree \( d \) elements \( f_n \in \Lambda(n) \) such that \( \pi_n(f_{n+1}) = f_n \) for all \( n \). A general element of \( \Lambda \) is a finite sum of degree \( d \) elements. Concretely, we can think of elements \( f \in \Lambda \) as bounded-degree power series in \( x_1, x_2, \ldots \) which are invariant under all permutations of the variables. The connection is that \( f_n = f(x_1, \ldots, x_n, 0, 0, \ldots) \).

Elements of \( \Lambda \) are called symmetric functions.

An example is given by the sequence of Schur polynomials \( s_{\lambda}(x_1, \ldots, x_n) \) (if \( \ell(\lambda) > n \), then this is defined to be 0), and the limit symmetric function is the Schur function \( s_{\lambda} \).

Note that \( \Lambda \) is also a ring under the usual addition and multiplication operations and that the specialization maps \( \Lambda \to \Lambda(n) \) obtained by \( x_{n+1} = x_{n+2} = \cdots = 0 \) are ring homomorphisms. Explicitly, this means that if we do computations, such as multiplication or plethysm, we can do it in the ring \( \Lambda \) and then we automatically get answers in \( \Lambda(n) \) for all \( n \). Heuristically: “the representation theory of polynomial representations of \( \text{GL}_n(\mathbb{C}) \) exhibits stability with respect to \( n \).” This will not hold for the symplectic and orthogonal groups in general, though will hold if we require \( n \gg 0 \).

1.4.4. **Littlewood–Richardson coefficients.**

1.5. **Flag varieties.**

1.5.1. **Grassmannians.**

1.5.2. **Borel–Weil theorem.**

2. **Symplectic groups**

3. **Orthogonal groups**

4. **Spin groups**

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