Bi-graded Koszul modules, K3 carpets, and Green’s conjecture

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Green’s conjecture

- Fix alg. closed field $k$ of characteristic 0.
- $C$ is a (smooth) genus $g \geq 2$ curve with canonical bundle $\omega_C$.
- The canonical ring $\Gamma_C = \bigoplus_{d \geq 0} H^0(C; \omega_C^d)$ is finitely generated over $A = \text{Sym} H^0(C; \omega_C) \cong k[x_1, \ldots, x_g]$.
- We’re concerned with vanishing of Betti numbers
  \[ \beta_{i,j}(C) = \dim_k \text{Tor}_i^A(\Gamma_C, k)_j. \]

- **Green’s conjecture** states that $\beta_{i,i+2}(C) = 0$ for $i < \text{Cliff}(C)$, the Clifford index of $C$. This governs for how many steps the equations of $C$ have only linear syzygies.
- For “most” curves, $\text{Cliff}(C) = \text{gon}(C) - 2$ where $\text{gon}(C)$ is the minimum degree of a non-constant map $C \to \mathbb{P}^1$. 
Voisin (2002, 2005): There is a nonempty Zariski dense set of curves in the moduli space of curves for which Green’s conjecture holds. Geometric proof involving K3 surfaces. Note: \( \text{Cliff}(C) \leq (g - 1)/2 \) for all \( C \)
Refinement: In fact, this set contains curves of each gonality.

Aprodu–Farkas (2011): Green’s conjecture holds for any curve that lies on a K3 surface.

Many other variations...

Aprodu–Farkas–Papadima–Raicu–Weyman (2019): Reproved Voisin’s result using representation theory ideas (next slide). Method of proof is simpler and extends to positive characteristic \( p \geq (g + 2)/2 \)

Schreyer (1986): Green’s conjecture fails in low characteristic
Rational cuspidal cubics

- Betti numbers are semicontinuous, i.e., the Betti numbers of a flat degeneration can only go up. Hence, to prove (generic) vanishing, it suffices to prove it for a single example, and we can consider singular (smoothable) curves.
- A rational curve with $g$ cusps has genus $g$ and can be realized as a hyperplane section of the tangential surface $T_g$ of the $g$-uple rational normal curve (= the union of its tangent lines).
- There is a short exact sequence of graded modules

$\begin{align*}
0 \to k[T_g] \to \tilde{k[T_g]} \to \omega_{k[P^1,\mathcal{O}(g)]}(-1) \to 0,
\end{align*}$

consisting of the homog. coordinate ring of $T_g$, its normalization, and the canonical module of the homog. coordinate ring of the $g$-uple RNC.
- The latter two can be understood, so it amounts to understanding a long exact sequence on Tor.
The problem reduces to showing that the following map is surjective for \( i \leq (g - 1)/2 \):

\[
\begin{align*}
\text{Tor}_{i+1}^A(k[T_g], k) & \xrightarrow{\sim} \text{Tor}_i^A(\omega_k[P^1, O(g)], k) \\
\bigwedge^{i+1}(\text{Sym}^{g-2} k^2) \otimes D^{2i}(k^2) & \bigwedge^i(\text{Sym}^{g-1} k^2) \otimes \text{Sym}^{g-2-i}(k^2)
\end{align*}
\]

The group \( SL_2(k) \) acts on everything in sight, so the map is equivariant. However, it is difficult to guess a formula.

There is more structure though: we fix \( i \) and sum over all \( g \). It turns out that both are f.g. modules over \( \text{Sym}(D^{i+1}k^2) \) and the sum of maps is linear.

The actual way forward is technical but insight comes from Hermite reciprocity isomorphism:

\[
\text{Sym}^n(D^m k^2) \cong \bigwedge^m(\text{Sym}^{m+n-1} k^2).
\]
Koszul modules

- The key to using the module structure on the sum is that the cokernel can be recast as a **Koszul module**.
- Given a subspace $K \subset \bigwedge^2 V$, the Koszul module $W(V, K)$ is the middle homology of the modified Koszul complex

$$\text{Sym } V \otimes K \rightarrow \text{Sym } V \otimes V(1) \rightarrow \text{Sym } V(2)$$

- In previous setting, $V = D^{i+1}k^2$ and $K = D^{2i}k^2$.
- AFPRW proved the following are equivalent:
  - $K^\perp \subset \bigwedge^2 V^*$ contains no nonzero rank 2 matrix
  - $W(V, K)$ is finite length
  - $W(V, K)_d = 0$ for all $d \geq \dim V - 3$

This is enough to prove Green’s conjecture for rational cuspidal curves, and hence for a nonempty dense subset of curves in the moduli space of curves.
• Double structures on $\mathbb{P}^1$ (ribbons) give a different degeneration of genus $g$ curves (intuition: if a non-hyperelliptic curve degenerates to a hyperelliptic one, this is the degeneration of the image of the canonical map)

• They are hyperplane sections of double structures on rational normal scrolls. More precisely, consider the projective space $P^g = P(Sym^a k^2 \oplus Sym^{g-1-a} k^2)$ with an $a$-uple RNC and $(g - 1 - a)$-uple RNC. Let $B$ be the homog. coordinate ring of the corresponding scroll.

• There is an extension

$$0 \to \omega_B \to B' \to B \to 0$$

where $B'$ is the homog. coordinate ring of a **K3 carpet**. It is a double structure on the scroll.
Differences and results

- The ribbons are smoothable to curves of gonality $a$. Hence proving Green’s conjecture for each ribbon proves it for most curves of each gonality (not just the maximal value ones).
- Can prove it holds in characteristic $p \geq a$. In generic case $a = (g - 1)/2$, this beats $(g + 2)/2$ from cuspidal curves and resolves a conjecture of Eisenbud–Schreyer.
- Coordinate ring $A$ of $\mathbf{P}(\text{Sym}^a k^2 \oplus \text{Sym}^{g-1-a} k^2)$ is bigraded.
- The syzygies of $\omega_B$ and $B$ are understood, so we again need to consider a long exact sequence. The problem reduces to showing that the following map is surjective for $i < a$:

$$
\begin{array}{ccc}
\text{Tor}_i^A(B, k)_{i+2} & \xrightarrow{\cong} & \text{Tor}_i^A(\omega_B, k)_{i+2} \\
\cong D^{i-1}k^2 \otimes & & \cong S^{g-3-i}k^2 \otimes \\
\bigwedge^{i+1}(S^{a-1}k^2 \oplus S^{g-2-a}k^2) & & \bigwedge^i(S^{a-1}k^2 \oplus S^{g-2-a}k^2)
\end{array}
$$
We can decompose the last map into bigraded components, fix them, and sum over all $a, g$. Again, both terms are finitely generated modules over a symmetric algebra and the cokernel is a bigraded Koszul module.

Given vector spaces $V_1, V_2$ and $K \subset V_1 \otimes V_2 \subset \bigwedge^2(V_1 \oplus V_2)$, $W(V, K)$ is the middle homology of

$$
\begin{align*}
\text{Sym}(V_1 \oplus V_2) \otimes K & \rightarrow \text{Sym}(V_1 \oplus V_2) \otimes V_1(0, 1) \\
\text{Sym}(V_1 \oplus V_2) \otimes V_2(1, 0) & \rightarrow \text{Sym}(V_1 \oplus V_2)
\end{align*}
$$

In previous setting, $V_1 = D^u k^2$, $V_2 = D^v k^2$, $K = D^{u+v-2} k^2 + D^{u+v} k^2$.

Raicu–Sam:

- $K^\perp \subset V_1^* \otimes V_2^*$ contains no nonzero rank $\leq 2$ matrix
- $W(V, K)_{d,e} = 0$ for $d, e \gg 0$
- $W(V, K)_{d,e} = 0$ for $d \geq \dim V_2 - 2$ and $e \geq \dim V_1 - 2$.

As before, this proves Green’s conjecture for ribbons.