1. (6 points) For both parts of this problem, suppose that a particle is moving back and forth on a straight line. The velocity of the particle at time t seconds is $v(t)$ centimeters per second, where:

$$v(t) = \frac{t}{2} - 1$$

(a) Find $\int_0^6 v(t)\,dt$, the particle's displacement between time 0 seconds and time 6 seconds.

We compute

$$\int_0^6 v(t)\,dt = \int_0^6 \frac{t}{2} - 1 \; dt$$

$$= \frac{t^2}{4} - t \Big|_0^6$$

$$= \left(\frac{36}{4} - 6\right) - \left(\frac{0^2}{4} - 0\right)$$

$$= 9 - 6 = \underline{\underline{3}}$$

Problem continued on next page

(b) Find $\int_0^6 |v(t)|\,dt$, the distance travelled by the particle between time 0 seconds and time 6 seconds.

$$|v(t)| = \left| t/2 - 1 \right| = \begin{cases} 1 - t/2 & \text{if} \quad t \leq 2 \\ t/2 - 1 & \text{if} \quad t \geq 2 \end{cases}$$

$$\int_0^6 |v(t)|\,dt = \int_0^2 |v(t)|\,dt + \int_2^6 |v(t)|\,dt$$

$$= \int_0^2 1 - t/2 \, dt + \int_2^6 t/2 - 1 \, dt$$

$$= t - t^2/4 \Big|_0^2 + t^2/4 - t \Big|_2^6$$

$$= \left(2 - \tfrac{4}{4}\right) - \left(0 - \tfrac{0}{4}\right) + \left(\tfrac{36}{4} - 6\right) - \left(\tfrac{4}{4} - 2\right)$$

$$= 1 - 0 + 3 - (-1)$$

$$= 5$$

2. (2 points) Calculate the derivative:

$$\frac{d}{dx}\left(\int_1^{1/x} \sin(3t)\,dt\right)$$

We ccmpute an antiderivative of $\sin(3t)$.

$$\int_a^b \sin(3t)\,dt = \frac{1}{3}\int_a^b 3\sin(3t)\,dt$$

$$= \frac{1}{3}\left(-\cos(3t)\right)\Big|_a^b$$

$$\int_1^{1/x} \sin(3t)\,dt = \frac{1}{3}\left(-\cos(3t)\right)\Big|_1^{1/x}$$

$$= \frac{-1}{3}\cos\left(\frac{3}{x}\right) + \frac{1}{3}\cos(3)$$

$$\frac{d}{dx}\left(\int_1^{1/x} \sin(3t)\,dt\right) = \frac{d}{dx}\left(\frac{-1}{3}\cos\left(\frac{3}{x}\right)\right) + \frac{d}{dx}\left(\cos(3)\right)$$

$$= \frac{d}{dx}\left(\frac{-1}{3}\cos\left(\frac{3}{x}\right)\right)$$

$$= \frac{-1}{3}\left(-\sin\left(\frac{3}{x}\right)\right) 3\left(-x^{-2}\right)$$

$$= -\frac{1}{3}\sin\left(\frac{3}{x}\right)$$

3. (4 points) Evaluate the integral. Please show all of your work.

$$\int t\cos(2t)\,dt$$

$$u = t \qquad v = \frac{1}{2}\sin(2t)$$

$$du = \cancel{dt}\ 1 \qquad dv = \cos(2t)$$

$$\int u\,dv = uv - \int v\,du$$

$$\int t\cos(2t)\,dt = t\cdot\frac{1}{2}\sin(2t) - \int \frac{1}{2}\sin(2t)\,dt$$

$$= \frac{t}{2}\sin(2t) - \frac{1}{2}\int \sin(2t)\,dt$$

$$= \frac{t}{2}\sin(2t) - \frac{1}{4}\int 2\sin(2t)\,dt$$

$$= \frac{t}{2}\sin(2t) - \frac{1}{4}\left(-\cos(2t)\right) + C$$

$$= \frac{t}{2}\sin(2t) + \frac{1}{4}\cos(2t) + C$$

4. (6 points)

(a) Find the partial fraction expansion (PFE) of the rational function:

$$\frac{2x^2 - 2x + 1}{(x-2)(x^2+1)}$$

$$\frac{2x^2 - 2x + 1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1}$$

$$\Rightarrow \quad 2x^2 - 2x + 1 = A(x^2+1) + (Bx + C)(x-2)$$

$$= Ax^2 + A + Bx^2 + Cx - 2Bx - 2C$$

$$= (A+B)x^2 + (C-2B)x + (A-2C)$$

$\Rightarrow$ Linear system of equations

$$\begin{aligned} 2 &= A + B \\ -2 &= -2B + C \\ 1 &= A - 2C \end{aligned}$$

$$\begin{aligned} 1 + 2C &= 2 - B \\ B &= 1 - 2C \\ -2 &= -2 + 4C + C \\ 0 &= 3C \end{aligned}$$

$$\Rightarrow \quad C = 0 \quad \Rightarrow \quad B = 1 \quad \Rightarrow \quad A = 1$$

$$\Rightarrow \quad \frac{2x^2 - 2x + 1}{(x-2)(x^2+1)} = \frac{1}{x-2} + \frac{x}{x^2+1}$$

Problem continued on next page

(b) Evaluate the integral:

$$\int \frac{2x^2 - 2x + 1}{(x-2)(x^2+1)}\, dx$$

$$\int \frac{2x^2 - 2x + 1}{(x-2)(x^2+1)}\, dx = \int \frac{1}{x-2}\, dx + \int \frac{x}{x^2+1}\, dx$$

$$= \int \frac{1}{x-2}\, dx + \frac{1}{2}\int \frac{2x}{x^2+1}\, dx$$

$$= \ln|x-2| + \frac{1}{2}\ln|x^2+1| + C$$

5. (4 points) Evaluate the integral. Please show all of your work.

$$\int \frac{1}{\sqrt{36 - x^2}}\, dx$$

$$x = 6\sin(\theta) \quad \Rightarrow \quad dx = 6\cos(\theta)\, d\theta$$

$$\int (36 - x^2)^{-\frac{1}{2}}\, dx$$

$$= \int \frac{6\cos(\theta)}{\sqrt{36 - 36\,\cancel{\cos}\,\sin^2(\theta)}}\, d\theta$$

$$= \int \frac{\cos(\theta)}{\sqrt{1 - \sin^2(\theta)}}\, d\theta = \int \frac{\cos(\theta)}{\sqrt{\cos^2(\theta)}}\, d\theta$$

$$= \int \frac{\cos(\theta)}{\cos(\theta)}\, d\theta = \int 1\, d\theta$$

$$= \theta + C$$

$$= \arcsin\left(\frac{x}{6}\right) + C$$

6. (4 points) Determine whether the improper integral converges or diverges. If it converges, find its value.

$$\int_4^5 \frac{1}{(x-4)^{5/2}}\,dx$$

This is an improper integral of Type II with a discontinuity at $x = 4$

$$\int_4^5 (x-4)^{-5/2}\,dx = \lim_{t \to 4^+} \int_t^5 (x-4)^{-5/2}\,dx$$

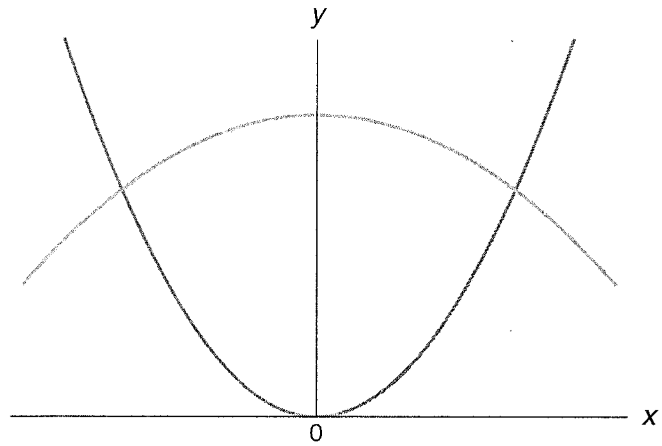
$$= \lim_{t \to 4^+} -\frac{2}{3}(x-4)^{-\frac{3}{2}} \Big|_t^5$$

$$= \lim_{t \to 4^+} -\frac{2}{3}(5-4)^{-3/2} + \frac{2}{3}(t-4)^{-3/2}$$

$$= -\frac{2}{3} + \lim_{t \to 4^+} \frac{2}{3}(t-4)^{-3/2}$$

The last term goes to infinity as t approaches 4,
Hence the improper integral is divergent.

7. (8 points) For both parts of this problem, let $\mathcal{R}$ be the region enclosed by the graphs of $y = 3x^2$ and $y = 4 - x^2$. In case you find it helpful, $\mathcal{R}$ is the shaded region in the figure below.



(a) Find the area of $\mathcal{R}$.

Intersection:

$$3x^2 = 4 - x^2$$
$$\Rightarrow \quad 0 = 4 - 4x^2$$
$$\Rightarrow \quad 4x^2 = 4$$
$$\Rightarrow \quad x = 1 \quad \text{or} \quad x = -1$$

$$A = \int_{-1}^{1} 4 - x^2 - 3x^2 \, dx$$
$$= \int_{-1}^{1} 4 - 4x^2 \, dx$$
$$= 4x - \frac{4}{3}x^3 \Big|_{-1}^{1}$$
$$= \left(4 - \frac{4}{3}\right) - \left(-4 + \frac{4}{3}\right)$$
$$= 4 + 4 - \frac{4}{3} - \frac{4}{3}$$
$$= 8 - \frac{8}{3}$$
$$= \frac{16}{3}$$

Problem continued on next page

(b) Find the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis.

$$V = \pi \int_{-1}^{1} (4 - x^2)^2 - (3x^2)^2 \, dx$$

$$= \pi \int_{-1}^{1} 16 - 8x^2 + x^4 - 9x^4 \, dx$$

$$= \pi \int_{-1}^{1} 16 - 8x^2 - 8x^4 \, dx$$

$$= \pi \left(16x - \frac{8}{3}x^3 - \frac{8}{5}x^5 \right) \Big|_{-1}^{1}$$

$$= \pi \left(\left(16 - 8/3 - 8/5\right) - \left(-16 + 8/3 + 8/5\right) \right)$$

$$= 2\pi \left(16 - 8/3 - 8/5 \right)$$

$$= 2\pi \cdot \left(\frac{240}{15} - \frac{40}{15} - \frac{24}{15} \right)$$

$$= \frac{2 \cdot 176}{15} \pi$$

$$= \frac{352}{15} \pi$$

8. (4 points) Find the solution of the differential equation that satisfies the given initial condition:

$$\frac{dy}{dx} = \frac{12x^2}{y}, \qquad y(1) = 3$$

Separable differential equation:

$$\frac{dy}{dx} = 12x^2 \cdot y = f(y) \cdot g(x)$$

with $f(y) = y^{-1}$, $g(x) = 12x^2$

$$\int \frac{1}{f(y)}\, dy = \int y\, dy = \frac{1}{2} y^2$$

$$\int g(x)\, dx = \int 12x^2\, dx = 4x^3$$

Solve

$$\frac{1}{2} y^2 = 4x^3 + C$$

$$y^2 = 8x^3 + 2C$$

$$y = \pm\sqrt{8x^3 + 2C}$$

Since $y(1)$ is positive, we use the positive solution and calculate

$$y(1) = 3 \quad \Rightarrow \quad 3 = \sqrt{8 \cdot 1 + 2C} = \sqrt{8 + 2C}$$

$$\Rightarrow \quad C = \frac{1}{2}$$

Solution:

$$y(x) = \sqrt{8x^3 + 1}$$

9. (4 points) Recall that the formula for the average value of the function f on the interval $[a, b]$ is:

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Use this formula to find the average value of the function $f(x) = 4xe^{x^2+3}$ on the interval $[0, 5]$.

$$f_{\text{ave}} = \frac{1}{5-0} \int_0^5 4x\, e^{x^2+3}\, dx$$

$$= \frac{2}{5} \int_0^5 2x\, e^{x^2+3}\, dx$$

$$= \frac{2}{5}\, e^{x^2+3} \Big|_0^5$$

$$= \frac{2}{5}\left(e^{5^2+3} - e^{0^2+3} \right)$$

$$= \frac{2}{5}\left(e^{28} - e^{3} \right)$$

10. (2 points) Determine whether the sequence converges or diverges. If it converges, find its limit.

$$a_n = \frac{7n^2 + 3n + 10}{5n^2 - 9}$$

The numerator and the denominator are polynomials of the same degree, namely degree 2.

The limit value is the ration of the leading coefficients

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{7n^2 + 3n + 10}{5n^2 - 9}$$

$$= \lim_{n \to \infty} \frac{7n^2}{5n^2} = \frac{7}{5}$$

11. (4 points) Determine whether the geometric series converges or diverges. If it converges, find its sum.

$$3 - \frac{6}{7} + \frac{12}{49} - \frac{24}{343} + \cdots$$

$a = 3, \quad r = -\frac{2}{7}$

$|r| < 1$, hence the geometric series converges

$$\sum_{n=1}^{\infty} 3\left(\frac{-2}{7}\right)^{n-1} = \frac{a}{1-r}$$

$$= \frac{3}{1 - \left(\frac{-2}{7}\right)}$$

$$= \frac{3}{\frac{7}{7} - \frac{-2}{7}}$$

$$= \frac{3}{\frac{7}{7} + \frac{2}{7}}$$

$$= \frac{3}{9/7}$$

$$= \frac{1}{3/7} = \frac{7}{3}$$

12. (4 points) Let $f(x) = \cos(x)$. Find the fourth degree Taylor polynomial $T_4(x)$ for f with center $a = 0$.

General formula

$$T_{4,a} f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \frac{f''''(a)}{24}(x-a)^4$$

$\boxed{a = 0}$

$$f(a) = \cos(0) = 1$$

$$f'(a) = -\sin(0) = 0$$

$$f''(a) = -\cos(0) = -1$$

$$f'''(a) = \sin(0) = 0$$

$$f''''(a) = \cos(0) = 1$$

$$T_{4,a} f(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$