Q1. Find the area of one petal of the rose given by

\[ r = 3 \sin (2\theta) \]
Answer Q1.

Step 1. Consider the graph of $r = 3\sin(2\theta)$ in Cartesian coordinates. All petals have the same area, so we pick the first one.

The first petal is traced when $\theta$ varies from 0 to $\frac{\pi}{2}$. 
Step 2. Integrate using area formula \( A = \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 \, d\theta \).

\[
A = \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} g \sin^2(2\theta) \, d\theta = \frac{g}{2} \int_0^{\frac{\pi}{2}} \sin^2(2\theta) \, d\theta
\]

Now recall: \( \sin^2(x) = \frac{1}{2} \left(1 - \cos(2x)\right) \); this implies

\[
A = \frac{g}{2} \int_0^{\frac{\pi}{2}} \frac{1}{2} \left(1 - \cos(4\theta)\right) \, d\theta = \frac{g}{4} \int_0^{\frac{\pi}{2}} 1 - \cos(4\theta) \, d\theta
\]

\[
= \frac{g}{4} \left( \theta - \frac{\sin(4\theta)}{4} \right) \bigg|_0^{\frac{\pi}{2}} = \frac{g}{4} \left[ \left( \frac{\pi}{2} - \frac{\sin(2\pi)}{4} \right) - (0 - \frac{\sin(0)}{4}) \right]
\]

\[
= \frac{g}{4} \cdot \frac{\pi}{2} = \frac{g \pi}{8}.
\]
Q2.

Write out the first 4 terms of the Taylor series of \( f(x) \) if

\[ f(5) = 9, \quad f'(5) = -2, \quad f''(5) = 0, \quad f'''(5) = 1. \]
Answer Q2.

We are looking for the Taylor series of \( f(x) \) centered at \( c = 5 \). Recall that this is given by

\[
T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!} (x-5)^n.
\]

The first 4 terms are

\[
f(5) + f'(5)(x-5) + \frac{f''(5)}{2!} (x-5)^2 + \frac{f'''(5)}{3!} (x-5)^3
\]

\[
= 9 + (-2) (x-5) + \frac{0}{2!} (x-5)^2 + \frac{1}{3!} (x-5)^3
\]

\[
= 9 - 2 (x-5) + \frac{1}{6} (x-5)^3.
\]
Q3. Find the integral

\[ \int \sin(\sqrt{x}) \, dx \]
We do $u$-sub together with integration by parts.

\[
\begin{align*}
  w &= \sqrt{x} \\
  dw &= \frac{1}{2} x^{-\frac{1}{2}} \, dx \\
  &= \frac{1}{2\sqrt{x}} \, dx \\
  &= \frac{1}{2w} \, dx \\
2w \, dw &= dx
\end{align*}
\]

\[
\int \sin(\sqrt{x}) \, dx = \int \sin(w) \, 2w \, dw
\]

\[
= 2 \int \frac{w \sin(w)}{w} \, dw
\]

\[
= 2 \int \sin(w) \, dw
\]

\[
v = -\cos(w) \\
\frac{dv}{dw} = \frac{1}{2w} \, dw = dw
\]

\[
= 2 \left( -w \cos(w) + \int \cos(w) \, dw \right)
\]

\[
= 2 \left( -w \cos(w) + \sin(w) \right) + C
\]

\[
= 2 \left( -\sqrt{x} \cos(\sqrt{x}) + \sin(\sqrt{x}) \right) + C
\]
Q4. Find the interval of convergence of

\[ \sum_{n=1}^{\infty} \frac{6^n (x+4)^n}{7^n} \]
Answer Q4. We apply the ratio test to $a_n = \frac{6^n (x+4)^n}{7n}$.

$$\rho = \lim_{n \to \infty} \left| \frac{\frac{6^{n+1} (x+4)^{n+1}}{7(n+1)}}{\frac{6^n (x+4)^n}{7n}} \right| = \lim_{n \to \infty} \left| \frac{6^{n+1} (x+4)^{n+1}}{6^n (x+4)^n} \cdot \frac{7n}{7(n+1)} \right|$$

$$= \lim_{n \to \infty} \left| 6 \cdot \frac{n}{n+1} (x+4) \right| = 6 |x+4| \lim_{n \to \infty} \frac{n}{n+1} = 6 |x+4|$$

Series converges if $6 |x+4| < 1 \Rightarrow |x+4| < \frac{1}{6}$

Convergence on $\left( -\frac{25}{6}, -\frac{23}{6} \right)$.
The series diverges if \( x < -\frac{25}{6} \) and if \( x > -\frac{23}{6} \).

We need to check the endpoints: \(-\frac{25}{6}, -\frac{23}{6}\).

1. \( x = -\frac{25}{6} \):

\[
\sum_{n=1}^{\infty} \frac{6^n (-\frac{1}{6})^n}{7n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{7n}
\]

\( b_n = \frac{1}{7n} \) is positive, decreasing and \( \lim_{n \to \infty} \frac{1}{7n} = 0 \).

By the Leibnitz test (alternating series test), the series converges.

2. \( x = -\frac{23}{6} \):

\[
\sum_{n=1}^{\infty} \frac{6^n (\frac{1}{6})^n}{7n} = \sum_{n=1}^{\infty} \frac{1}{7n}
\]

This is a harmonic series \( \to \) divergent!
Full answer: The interval of convergence is \([-\frac{23}{6}, -\frac{23}{6})\).

The series diverges if \(x < -\frac{25}{6}\) and if \(x \geq -\frac{23}{6}\).
Q5. Does the following series converge or diverge?

\[ \sum_{n=1}^{\infty} \left( \frac{5+n}{n} \right)^n \]
Answer Q5.

Apply divergence test (nth term test):

$$\lim_{n \to \infty} \left( \frac{5+n}{n} \right)^n = \lim_{n \to \infty} \left( \frac{5}{n} + 1 \right)^n = \lim_{n \to \infty} \left( 1 + \frac{5}{n} \right)^n = e^5 \neq 0.$$  

Since the limit is not 0, the series diverges.

Note: The root test is inconclusive:

$$\lim_{n \to \infty} \sqrt[n]{\left| \frac{5+n}{n} \right|^n} = \lim_{n \to \infty} \frac{5+n}{n} = \lim_{n \to \infty} \frac{5}{n} + 1 = 1 \rightarrow \text{inconclusive}.$$
Note: It is good to remember this limit:

$$\lim_{{n \to \infty}} \left(1 + \frac{c}{n}\right)^n = e^c$$

for a constant $c$. 
Q6. Does the following series converge or diverge?

\[ \sum_{n=1}^{8} \frac{7}{n^2 + 1} \]
Answer Q6.

Step 1. Divergence test: \( \lim_{n \to \infty} \frac{7}{n^2 + 1} = 0 \Rightarrow \text{inconclusive.} \)

Step 2. Since the series is positive, we can apply all test available for positive series.

Step 3. Use integral test. \( f(x) = \frac{7}{1 + x^2} \) is positive, decreasing and continuous for \( x \geq 1 \).

Compute \( \int_1^\infty \frac{7}{1 + x^2} \, dx = \lim_{R \to \infty} \int_1^R \frac{7}{1 + x^2} \, dx = \lim_{R \to \infty} 7 \arctan(x) \)
\[ 7 \lim_{R \to \infty} \arctan(R) - \arctan(1) = 7 \left( \frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{7\pi}{4} \]

Since the integral converges, also the series converges.

Note: Remember that \( \lim_{x \to \infty} \arctan(x) = \frac{\pi}{2} \)
\( \lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2} \).
Q7. Find the volume $V$ of the solid obtained by rotating the region under the graph of the function $f(x) = \frac{2}{x+1}$ about the $x$-axis over the interval $[0,3]$. 
Answer Q7. The volume of the solid is

\[ V = \pi \int_0^3 \left( \frac{2}{x+1} \right)^2 \, dx = 4\pi \int_0^3 \frac{1}{(x+1)^2} \, dx = 4\pi \frac{(x+1)^{-1}}{-1} \bigg|_0^3 \]

\[ = -4\pi \left( \frac{1}{3+1} - \frac{1}{0+1} \right) = -4\pi \left( \frac{1}{4} - 1 \right) = -4\pi \left( -\frac{3}{4} \right) \]

\[ = 3\pi \]