Four Facts on 4-manifolds for Food 4 Thought

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UNIVERSITY OF CALIFORNIA · SAN DIEGO · LET THERE BE LIGHT

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What to expect:

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How topologists do it?

Making "weaker" equals signs and topological invariants to tell spaces apart

Some Equivalences

Equivalences

Some Equivalences

Equivalences

Homotopy $\boxed{\simeq}$

Some Equivalences

Equivalences

Homotopy $\boxed{\simeq}$ | Homeomorphism $\boxed{\cong}$

Some Equivalences

Equivalences

Homotopy $\boxed{\simeq}$ | Homeomorphism $\boxed{\cong}$ | Diffeomorphism $\boxed{\cong}$

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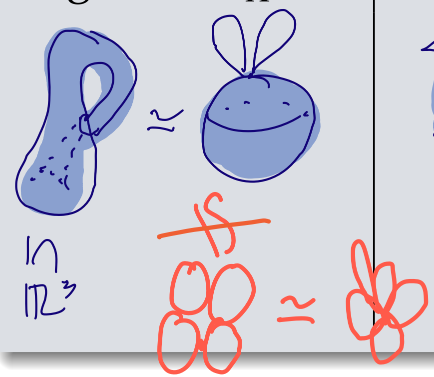
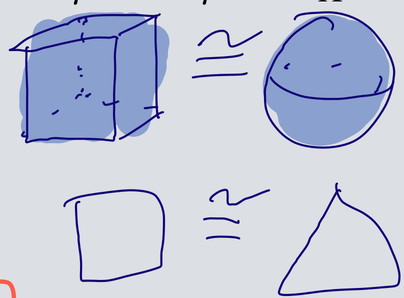
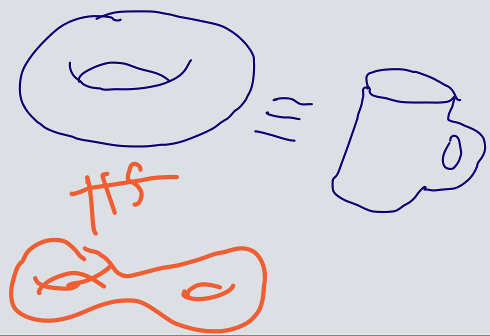
Equivalences

Homotopy $\simeq$	Homeomorphism $\cong$	Diffeomorphism $\cong$
Cts. Deform	Cts. Bijection + inv.	Smth Bijection + inv.
$X \simeq Y$ iff	$X \underset{\text{homeo.}}{\cong} Y$ iff	$X \underset{\text{diff.}}{\cong} Y$ iff
$X \underset{g}{\overset{f}{\rightleftarrows}} Y$ cts.	$X \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} Y$ cts.	$X \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} Y$ smth.
$f \circ g \simeq 1_Y,$ $g \circ f \simeq 1_X$	$\varphi \circ \varphi^{-1} = 1_Y$ $\varphi^{-1} \circ \varphi = 1_X$	$\varphi \circ \varphi^{-1} = 1_Y$ $\varphi^{-1} \circ \varphi = 1_X$

Some Invariants

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(Co)Homology $\boxed{H_*(X;\mathbb{Z})}$

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(Co)Homology $H_*(X;\mathbb{Z})$	Fundamental Groups $\pi_1(X, x_0)$
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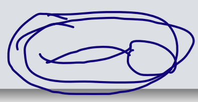
(Co)Homology $H_*(X;\mathbb{Z})$	Fundamental Groups $\pi_1(X, x_0)$
"Counts Holes"	"Counts loops"
$H_*(\mathbb{S}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else} \end{cases}.$	$\pi_1(\mathbb{S}^1, 1) = \mathbb{Z}$ $\pi_1(\mathbb{S}^n, e_1) = 0 \quad n \geq 2$
$H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ where $\alpha \in H^2(\mathbb{CP}^n;\mathbb{Z})$	$\pi_1(\mathbb{RP}^1, [e_1]) = \mathbb{Z}$ $\pi_1(\mathbb{RP}^n, [e_1]) = \mathbb{Z}/2\mathbb{Z} \quad n \geq 2$
$H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1})$ $x \in H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$	$\pi_1(\mathbb{T}^2, \vec{1}) = \mathbb{Z} \oplus \mathbb{Z}$
$H^*(\mathbb{T}^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & * = 1 \\ 0 & \text{else} \end{cases}$	$\pi_1(\mathbb{CP}^n, [e_1]) = 0$



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Dimension

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"How many indpendent pieces of data you need to describe a point in your space."

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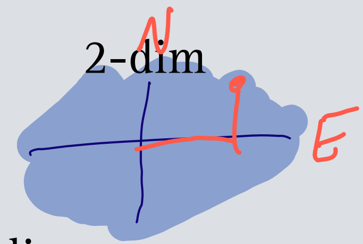
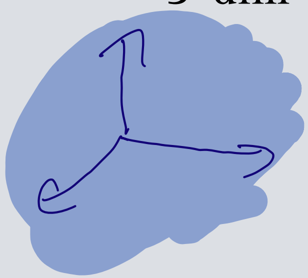
0-dim

1-dim

2-dim



3-dim



4-dim

Dust.

Manifold

Manifold

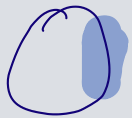
Manifold

Manifold

Locally looks like $\mathbb{R}^n$

0-dim

1-dim



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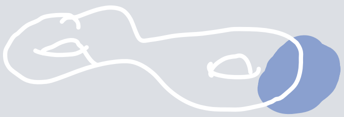

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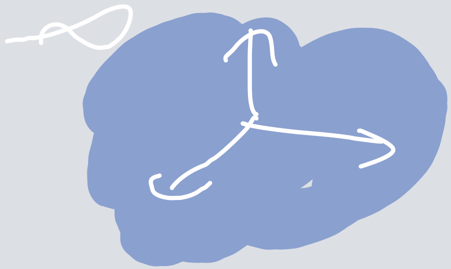

4-dim

$\mathbb{C}P^2$ $\quad K3$

$S^2 \times S^2$ $\quad S^1 \times S^3$

$\mathbb{R}P^4$ $\quad Gr_2(4)$

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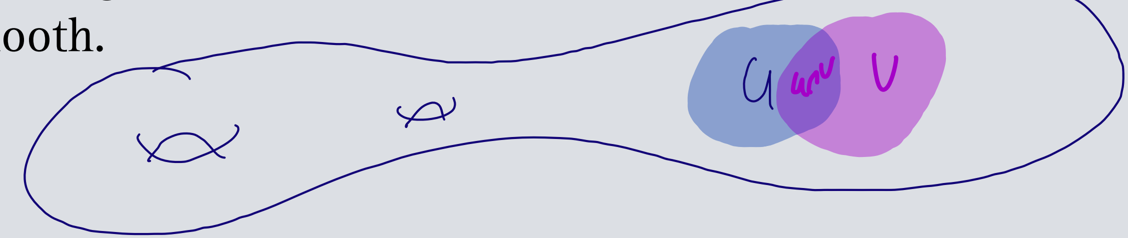

$$\begin{array}{ccc} U \cap V & \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} & U \cap V \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \mathbb{R}^n \cong U & & V \cong \mathbb{R}^n \end{array}$$

A **smooth structure** is a maximal atlas (local neighborhoods and charts that are smoothly compatible) on the smooth manifold.

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Theorem (FACT #1: Markov, 60s)

Given a finitely presented group $G := \langle g_1, \ldots, g_n \mid r_1, \ldots, r_k \rangle$, there exists a smooth, closed 4-manifold X whose fundamental group $\pi_1(X, x_0) = G$.

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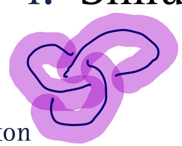

$\partial(S^1 \times B^3) = \partial(B^2 \times S^2)$

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3. Each r_i can be rep'd by nonintersecting embedded submanifolds $[\gamma_i] \in \pi_1$ by transversality.
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   $\pi_1(\text{surgery}(S^1 \times S^3)^{\# n}, 1) = \langle g_1, \dots, g_n \mid r_1, \dots, r_n \rangle$.

$S^1 \times B^2 \subset M^3$

$B^2 \times S^1$

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$$\begin{array}{ccccc} H_0(M;\mathbb{Z}) & H_1(M;\mathbb{Z}) & H_2(M;\mathbb{Z}) & H_3(M;\mathbb{Z}) & H_4(M;\mathbb{Z}) \\ \| & \| & \| & \| & \| \\ \mathbb{Z} & 0 & \mathbb{Z}^r & 0 & \mathbb{Z} \\ \| & \| & \| & \| & \| \\ H^0(M;\mathbb{Z}) & H^1(M;\mathbb{Z}) & H^2(M;\mathbb{Z}) & H^3(M;\mathbb{Z}) & H^4(M;\mathbb{Z}) \end{array}$$

Sketch of Proof.

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- Poincaré duality gives $H_2(M;\mathbb{Z}) = H^2(M;\mathbb{Z})$ and UCT and M closed mfld. gives $H_2(M;\mathbb{Z}) = \mathbb{Z}^r$.

Big Theorem

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Given any integral, unimodular, symmetric bilinear form Q, there exists a topological, simply connected closed 4-manifold M with $Q_M = Q$.
If $Q(\alpha, \alpha) = 0 \mod 2$ for all α, then M is unique up to homeomorphism.
Otherwise, there are two homeomorphism types of such an M, and at most one admits a smooth structure.

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Sketch of Proof.
Fields Medal worthy.

UC San Diego

Some Definitions

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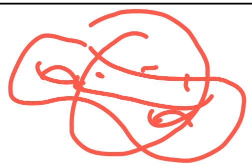
$H_2(M;\mathbb{Z}) = \mathbb{Z}^r = H^2(M;\mathbb{Z})$, and we have cup product. The intersection form Q_M is:

$$\begin{array}{ccc}
H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) & \xrightarrow{Q_M} & \mathbb{Z} \\
\Big\| \text{ P.D} & & \Big\uparrow \langle -, [M] \rangle \\
H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) & \xrightarrow{\smile} & H^4(M;\mathbb{Z})
\end{array}$$

Algebraic	Geometric
	

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Theorem (FACT #3: Freedman, 1982)

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- For more rigor read Manolescu's 4.11 section.

References

- Ciprian Manolescu's notes on 4-manifolds `https://stanford.edu/ sfh/283A.pdf`

- Freedman, Michael Hartley. "The topology of four-dimensional manifolds." Journal of Differential Geometry 17.3 (1982): 357-453.

- Tu, Loring W, and Springerlink (Online Service. An Introduction to Manifolds. New York, Ny, Springer New York, 2011.

$$\mathcal{T}\mathbb{H}\alpha\mathbb{N}\kappa$$

$$\text{YO}_{(4)}\text{U}_{(4)}!!!!$$

What I got for 1-44:

1	$\frac{4}{4}\cdot\frac{4}{4}$	12	$\frac{4!}{\sqrt{4}}\cdot\frac{4}{4}$	23	$4!-\frac{4}{\sqrt{4}\sqrt{4}}$	34	$4!+4+4+\sqrt{4}$
2	$\frac{4}{4}+\frac{4}{4}$	13	$\frac{4!}{\sqrt{4}}+\frac{4}{4}$	24	$4!\cdot\left(\frac{4}{4}\right)^4$	35	next page
3	$\frac{4}{4}+\frac{4}{\sqrt{4}}$	14	$\frac{4!}{\sqrt{4}}+\frac{4}{\sqrt{4}}$	25	$4!+\left(\frac{4}{4}\right)^4$	36	$4!\sqrt{4}-\frac{4!}{\sqrt{4}}$
4	$\frac{4}{\sqrt{4}}+\frac{4}{\sqrt{4}}$	15	$4\cdot 4-\frac{4}{4}$	26	$4!+\frac{4+4}{4}$	37	next page
5	$\sqrt{4\cdot 4}+\frac{4}{4}$	16	$4\cdot 4\cdot\frac{4}{4}$	27	$4!+\frac{4!}{4+4}$	38	$44-4-\sqrt{4}$
6	$\frac{4!\cdot 4}{4\cdot 4}$	17	$4\cdot 4+\frac{4}{4}$	28	$4!+4\cdot\frac{4}{4}$	39	next page
7	$\frac{4!}{4}+\frac{4}{4}$	18	$4\cdot 4+\frac{4}{\sqrt{4}}$	29	$4!+4+\frac{4}{4}$	40	$4!\sqrt{4}-4-4$
8	$(4+4)\cdot\frac{4}{4}$	19	$4!-4-\frac{4}{4}$	30	$4!+\frac{4!}{\sqrt{4\cdot 4}}$	41	next page
9	$4+4+\frac{4}{4}$	20	$(4!-4)\cdot\frac{4}{4}$	31	$4!+\frac{4!}{4}+\lfloor\sqrt{\sqrt{4}}\rfloor$	42	$44-\frac{4}{\sqrt{4}}$
10	$4+4+\frac{4}{\sqrt{4}}$	21	$4!-4+\frac{4}{4}$	32	$4!+\sqrt{4\cdot 4\cdot 4}$	43	$44-\frac{4}{4}$
11	$\frac{4!}{4}-\frac{4}{4}$	22	$4!-4+\frac{4}{\sqrt{4}}$	33	$4!+4+4+\lfloor\sqrt{\sqrt{4}}\rfloor$	44	$4!\sqrt{4}-\sqrt{4\cdot 4}$

Extra From The owner of my right shoe

$$35 = \left\lfloor \sqrt{\sqrt{\sqrt{\sqrt{(4!)!}}} + 4! + 4} \right\rfloor \cdot \lfloor \sqrt{\sqrt{4}} \rfloor$$

$$37 = \left\lfloor \sqrt{\sqrt{\sqrt{\sqrt{(4!)!}}} + 4! + 4} \right\rfloor + \sqrt{4}$$

$$39 = \left\lfloor \sqrt{\sqrt{\sqrt{\sqrt{(4!)!}}} + 4! + 4} \right\rfloor + 4$$

$$41 = 44 - \lceil \sqrt{4!} \rceil + \sqrt{4}$$