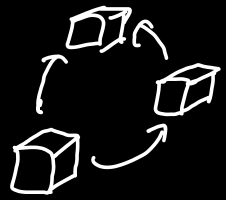
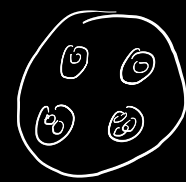


E_n



Operads 101A

$(k; n_1 \cdots n_k)$

$\downarrow$

$\sum_{i=1}^{k} n_i$

Scotty T.

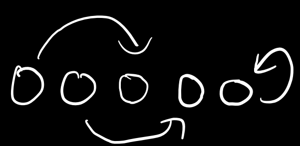
eCHT REU

June 30, 2023

E_∞

A_n

$A_\infty = E_1$



$\mathcal{O}(0) = \{*\}$

Purpose | Introduce you to operads

Successful Outcome | (a) You can write the abstract definition of an operad

(b) You know 3 examples

(c) I make some words stick in your head

Plan |
- Motivate
- Define an operad
- Example
- Add adjectives to operads "operadjectives?!"

Motivate

Rough estimates

Elementary school $4 =$  $\triangle$ = triangle

Middle school $x = 7$

High school $f'(x) = 0$

College $\forall \varepsilon > 0 \ \exists \delta$ s.t. $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$

$3 \equiv 7 \bmod 4$ $\Rightarrow \lim_{x \to y} f(x) = f(y)$.

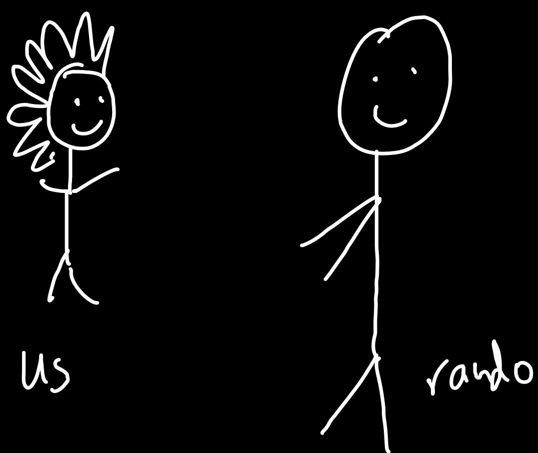
Grad school $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$

$\bigcirc = \triangle$

$G/\ker\varphi \cong \operatorname{Im}(\varphi)$

Math is all about "=" and what it means.

Scenario

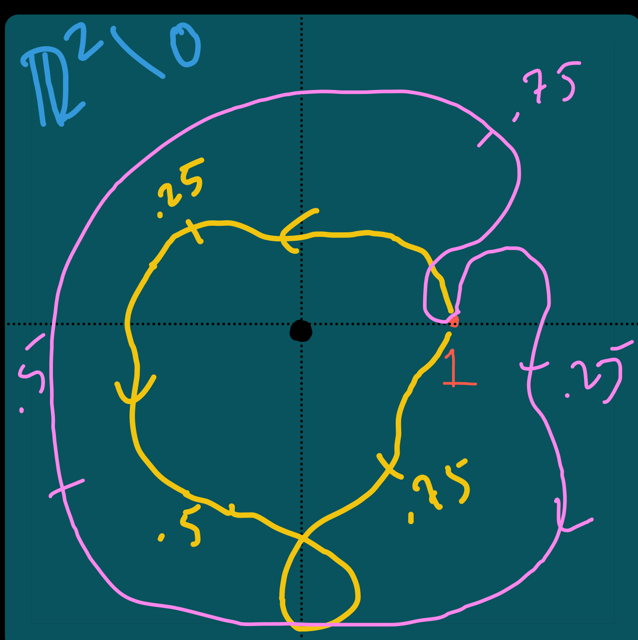


R: Math? Do you love multiplication?

U: Yeah!

R: Impress me.

Loops



$\alpha : [0,1] \to \mathbb{R}^2 \setminus 0$

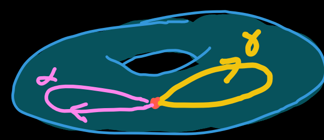
$\gamma : [0,1] \to$

$$\alpha * \gamma = \begin{cases} \alpha(2t) & t \in [0, \tfrac{1}{2}] \\ \gamma(2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

concatenation "multiplying" loops

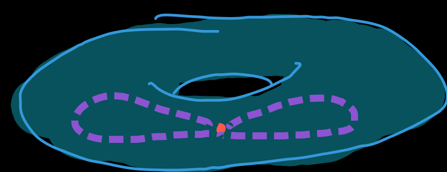
Def A loop in X based at x a continuous map $\gamma : [0,1] \to X$ where $\gamma(0) = \gamma(1) = x$

$$\gamma : ([0,1], 1) \longrightarrow (X, x) \qquad \gamma(0) = \gamma(1)$$



$$\alpha : ([0,1], 1) \longrightarrow (X, x) \qquad \alpha(0) = \alpha(1)$$

$$\alpha * \gamma : ([0,1], 1) \longrightarrow (X, x)$$



Consider Paths $\alpha, \beta, \gamma : [0,1] \to X$

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & t \in [0, \tfrac{1}{2}] \\ \beta(2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases} \qquad \beta * \gamma(t) = \begin{cases} \beta(2t) & t \in [0, \tfrac{1}{2}] \\ \gamma(2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

$$(\alpha * \beta) * \gamma(t) = \begin{cases} \alpha(4t) & t \in [0, \tfrac{1}{4}] \\ \beta(4t-1) & t \in [\tfrac{1}{4}, \tfrac{1}{2}] \\ \gamma(2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases} \qquad \alpha * (\beta * \gamma)(t) = \begin{cases} \alpha(2t) & t \in [0, \tfrac{1}{2}] \\ \beta(4t-2) & t \in (\tfrac{1}{2}, \tfrac{3}{4}] \\ \gamma(4t-3) & t \in [\tfrac{3}{4}, 1] \end{cases}$$

$$(\alpha * \beta) * \gamma(t) \underset{\text{functions}}{\overset{\text{as}}{\neq}} \alpha * (\beta * \gamma)(t)$$:(too fine

$$\mathrm{Im}\big((\alpha * \beta) * \gamma\big) \underset{\text{sets}}{=} \mathrm{Im}\big(\alpha * (\beta * \gamma)\big)$$:(too coarse

What a bout Homotopy

Defn - full - general Let $f, g : X \longrightarrow Y$ be two continuous maps. We say f is homotopic to g i.e. $f \simeq g$ if there exists a homotopy

$$H : \underset{\text{space}}{X} \times \underset{\text{time in youtube clip}}{[0,1]} \xrightarrow{\text{cts}} Y$$

such that

$$H(x,0) = f(x) \text{ and } H(x,1) = g(x).$$

etymology

homo - topy
" "
"same" "place"

Scotty ology: A homotopy is a math movie that takes 1 second to move images to the same.

You're New Best friend Linear interpolation. Let $x, y \in \mathbb{R}^n$. The line $\ell(t) := (1-t)x + ty$ is a straight line from x to y.

$$H(t,s) = \begin{cases} \alpha\big(2t(1-s) + 4t(s)\big) & t \in [0, \frac{1}{2} - \frac{1}{4}s] \\ \beta\big((4t-2)(1-s) + (4t-1)(s)\big) & t \in [\frac{1}{2} - \frac{1}{4}s, \frac{3}{4} - \frac{1}{4}s] \\ \gamma\big((4t-3)(1-s) + (2t-1)(s)\big) & t \in [\frac{3}{4} - \frac{1}{4}s, 1] \end{cases}$$

$$H(t,0) = \begin{cases} \alpha(2t) & t \in [0, \frac{1}{2}] \\ \beta(4t-2) & t \in [\frac{1}{2}, \frac{3}{4}) \\ \gamma(4t-3) & t \in (3/4, 1] \end{cases} \qquad H(t,1) = \Big\{$$

So the "$=$" we'll look for is "$\simeq$".

But, ugh, I want to be clear when I write $\alpha * \beta * \gamma * \delta$, but we know from Catalan club that

$$(\alpha\beta)(\gamma\delta), \quad ((\alpha\beta)\gamma)\delta$$
$$(\alpha(\beta\gamma))\delta \quad \alpha(\beta(\gamma\delta))$$
$$\alpha((\beta\gamma)\delta).$$

operations

Let $\mathcal{O}(4) =$ Spaces of homotopies from each of these to each other.

added structure

or

$$= \text{Embeddings}\big((0,1) \sqcup (0,1) \sqcup (0,1) \sqcup (0,1) \to (0,1)\big)$$

We would like a few things to work out

Say we have

$$\underset{\text{2 functions}}{\mathcal{O}(2)} \times \underset{3}{\mathcal{O}(3)} \times \underset{4.}{\mathcal{O}(4)}.$$

It should work out

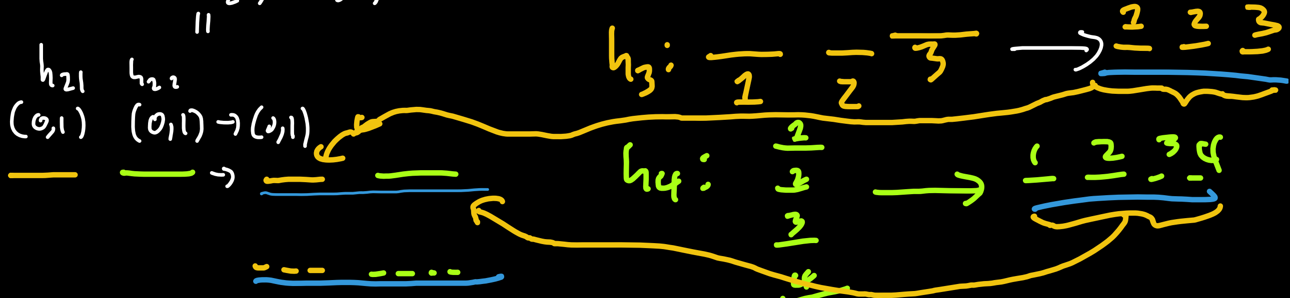
so then for

$$(h_2, h_3, h_4) \mapsto h_{21}(h_3) \sqcup h_{22}(h_4) \in \mathcal{O}(7)$$

$h_2 = $

h_{21} h_{22}

$(0,1)$ $(0,1) \to (0,1)$



Real Life Operad Definition

Def^n An **operad** is

- A sequence $\{\mathcal{O}(n)\}_{n\geq 0}$ of spaces where $\mathcal{O}(0)=*$

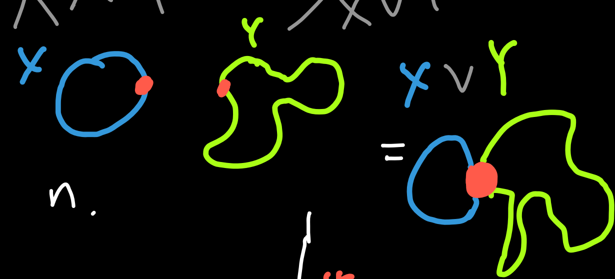
- For each k, for any $n_1, n_2, \dots, n_k$, maps
  $\gamma : \mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \longrightarrow \mathcal{O}(\sum n_k)$
- A specific element $\mathbb{1} \in \mathcal{O}(1)$
- An action $\Sigma_n \curvearrowright \mathcal{O}(n)$

Satisfying

- $\gamma(\gamma(k; n_1, \dots n_k); m_1 \cdots m_{\sum n_i}) = \gamma(k; \gamma(n_1; m_1 \dots m_{n_1}), \gamma(n_2; m_{n_1+1} \cdots m_{n_1+n_2}), \dots \gamma(n_k; \dots))$

Examples

$X \wedge Y \cong X \times Y / X \vee Y$



- $\mathcal{O}(n) = *$ for all n.

- $\mathcal{O}(n) = \mathrm{Maps}_*(X^{\wedge n}, X)$

- $\mathcal{O}(n) = \mathrm{Embeddings}^{or}((\mathbb{D}^k)^{\amalg n}, \mathbb{D}^k)$

  we saw this for $k = 1$

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$$\gamma(\underbrace{\gamma(2;1,3)}_{\mathcal{O}(4)};\ 4,5,11,19) \in \mathcal{O}(39)$$

$$\parallel$$

$$\gamma(2;\ \underbrace{\gamma(1;4)}_{\mathcal{O}(4)},\ \underbrace{\gamma(3;5,11,19)}_{\mathcal{O}(35)}) \in \mathcal{O}(39)$$

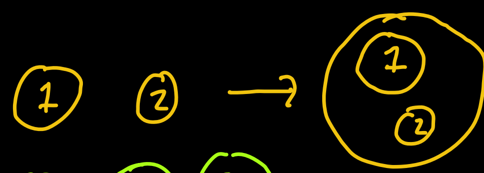
$$\gamma : \mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \mathcal{O}(n_k) \to \mathcal{O}\left(\sum n_i\right)$$

Example, Fleshed Out

$\mathcal{O}(3) = \mathrm{Emb}^{or}((\mathbb{D}^2)^{\sqcup 3}, \mathbb{D}^2)$

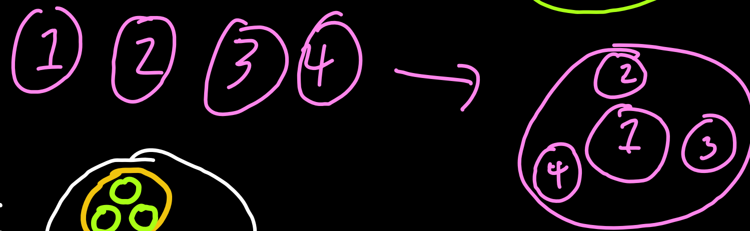
$\gamma : \mathcal{O}(2) \times \mathcal{O}(3) \times \mathcal{O}(4) \to \mathcal{O}(7)$

$\mathcal{O}(2) \ni f :$

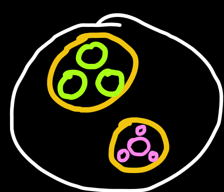


$\mathcal{O}(3) \ni g :$

$\mathcal{O}(4) \ni h :$

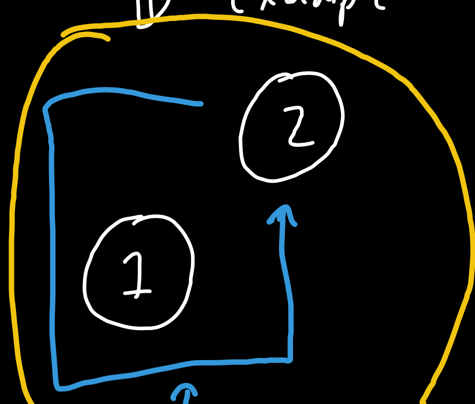


$\mathcal{O}(7) \ni \gamma(f; g, h) :$



Differences

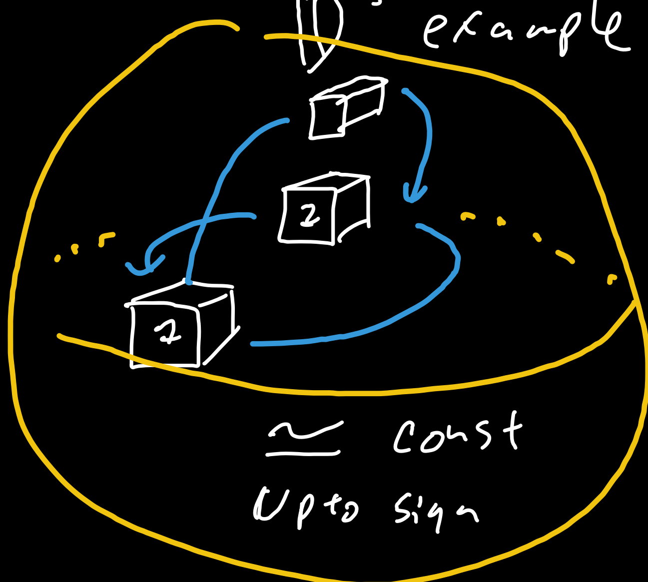
$\mathbb{D}^2$ example



path in $\mathcal{O}(2)$

$\not\simeq$ const.

$\mathbb{D}^3$ example



Adjectives

An operad is an

Associative A_n operad if $\mathcal{O}(k) \simeq *$ for all $k \leq n$

A_∞ operad if $\mathcal{O}(k) \simeq *$ for all k

E_1 ''

E_n operad if Σ_k action is free up to n

Assoc. + Comm. > Everything E_∞ operad if Σ_k action is free for all k

---

More Examples

Let X be a space and consider

$\Omega X := \mathrm{Loops}_x([0,1], X)$

Fact $\Omega^n X \cong \mathrm{Maps}_{N,x}(S^n, X)$

Fact $\Omega^n X$ have a great E_n operad given by $\mathrm{Emb}^{or}_{\partial}((\mathbb{D}^n)^{\sqcup k}, \mathbb{D}^n)$

Recap

- Goldilocks "=" for $*$ is $\simeq$
- operads contain all the info to describe this "$\simeq$" so we don't lose info
- Sometimes we get associative or assoc. + comm. = $\underline{E}$verything
- Disks in disks is a great example

When We Return

G - Spaces
N_∞ - operads
Results
Examples

Exercise

(2) Come up with a question for me to address next time and message me on Zulip.

References
or More Reading

(1) Higher Operads — Notes by Fiore, Ormsby

(2) Lectures on Factorization Homology, ∞-Categories, and Topological Field Theories — Tanaka

(3) Homotopy Everything H-spaces — Boardman, Vogt

(4) Handbook of Homotopy Theory — Lawson

(5) A small catalogue of E_n operads — Beuckelmann, Moerdijk

(6) Multiplicative Structures induced by N_∞ operads — Zhang