$\operatorname{colim}_n \pi_{n+k} X$

Σ Ω

Introduction To SPECTRA

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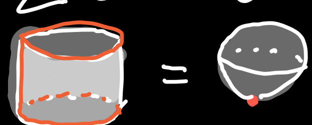
$$\begin{tikzcd} \mathrm{Top}_* \arrow[r, shift left, "\Sigma^\infty"] \arrow[rr, bend left=60, "\Sigma^\infty"] & \mathrm{PreSp} \arrow[l, shift left, "\Omega^\infty"] \arrow[r, shift left, "L"] & \mathrm{Sp} \arrow[l, shift left, "u"] \arrow[ll, bend left=60, "\Omega^\infty"] \end{tikzcd}$$

A few things to Memorize before we go forward

- We're working in Top_* when we talk about topological spaces. 
- the reduced suspension of a space

$$\Sigma X \cong S^1 \wedge X \cong X \times [0,1] \Big/ \begin{smallmatrix} X\times\{0\} \\ X\times\{1\} \\ \{*\}\times[0,1] \end{smallmatrix}$$

Ex) $\Sigma S^1 = S^2$



- the loop space $\Omega X = \mathrm{Maps}^{\mathrm{ces}}(S^1, X)$

and these are adjoint. i.e.

a map $\Sigma X \to Y$ has the same info as a map $X \to \Omega Y$

What are Spectra Good For?

mini **Definition** a (pre) spectrum T

- a sequence of spaces $T = \{T_n\}_{n \geq 0}$
- maps $\Sigma T_n \longrightarrow T_{n+1}$

$T_n \to \Omega T_{n+1}$

Satisfying some compatibility properties

mini **Example** $\mathbb{S}^0 = \{S^0, S^1, S^2, \dots\}$

w/ $\Sigma S^n \xrightarrow{\cong} S^{n+1}$

They are good for:

1. Generalized (co)homology

2. Stable Properties

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1 Generalized (co)homology

Def$^{\underline{n}}$ a generalized relative homology theory is

- Functors E_n : homotopy category of CW-pairs of spaces $\longrightarrow$ Abelian Groups for each n

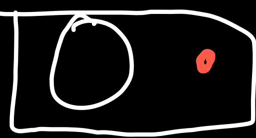
- natural transformations $E_n(X,A) \xrightarrow{\partial} E_{n-1}(A)$

satisfying $A \hookrightarrow X \hookrightarrow (X,A)$

- Exactness $\cdots \to E_q(A) \to E_q(X) \to E_q(X,A) \xrightarrow{\partial} E_{q-1}(A) \to \cdots$
- Excision $X = A \cup B$, $(A, A\cap B) \hookrightarrow (X,B) \Rightarrow E_*(A, A\cap B) \xrightarrow{\cong} E_*(X,B)$
- Additivity $(X,A) = \coprod_i (X_i, A_i) \Rightarrow \bigoplus_i E_*(X_i, A_i) \xrightarrow{\cong} E_*(X,A)$

For the (co) make the functors be contravariant.

A reduced homology is defined by $\tilde{E}_q(X) = E_q(X,*)$.

$S^1_+ =$  $\qquad \tilde{E}_q(X_+) = E_q(X)$

$X_+ = X \sqcup \{*\}$

Theorem $\tilde{H}^n(X;G) \cong [X, K(G,n)]$

$K(G,n)$ is an Eilenberg Mac Lane Space, meaning

$$\pi_j(K(G,n)) = \begin{cases} G & j=n \\ 0 & j\neq n, j\geq 1 \end{cases}$$

Exs S^1 is a $K(\mathbb{Z},1)$ $\quad \mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2,1)$

$\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$ $\quad L(\infty,p) = S^\infty/(\mathbb{Z}/p)$ is a $K(\mathbb{Z}/p,1)$.

Theorem Let $T = \{T_n\}$ be an Ω-prespectrum.

Define
$$\tilde{E}^q(X) = \begin{cases} [X, T_q] & q \geq 0 \\ [X, \Omega^{|q|} T_0] & q < 0. \end{cases}$$

Then $\{\tilde{E}^q\}$ is a reduced cohomology theory.

We aren't even at spectra yet!!!

Example | Complex K-Theory

Note $U(n) = \{A \in Mat_{n\times n}(\mathbb{C}) \mid A^*A = \mathbb{1}\}$

$$U := \operatorname{colim}\left(U(1) \hookrightarrow U(2) \hookrightarrow \cdots\right)$$

$EU(n) = \{(e_1, \dots, e_n) \mid e_i \cdot e_j = \delta_{ij}\}$ is the total space

$BU(n) = EU(n)/U(n)$ is its classifying space.

$$BU := \operatorname{colim}\left(BU(1) \xrightarrow{i_1} BU(2) \xrightarrow{i_2} \cdots\right)$$

Complex K-Theory ct'd

Note $\Omega U \simeq BU \times \mathbb{Z}$

$\Omega BU \times \mathbb{Z} \simeq U$

$\Rightarrow BU \times \mathbb{Z}, U, BU \times \mathbb{Z}, \dots$ is an $\underline{\Omega\text{-prespectrum}}$

so for X compact

$$\tilde{K}^0(X) = [X, BU \times \mathbb{Z}] = \tilde{K}^{2n}(X) \qquad \tilde{K}^1(X) = [X, U] = \tilde{K}^{2n+1}(X)$$

stable vector bundles

This gives us info about vector bundles over X and this is a generalized cohomology theory.

② Stable Properties

Freudenthal Suspension Theorem

Let X be $(n-1)$-connected. Note that we have maps
$$\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}\Sigma X.$$
Then when $q < 2n-1$ Σ is a bijection, and when $q = 2n-1$, it is a surjection.

This allows us to define the stable homotopy groups
$$\pi_k^{s.t}(X) := \operatorname*{colim}_n \pi_{n+k}(\Sigma^n X).$$

$$\operatorname*{colim}_n \left(\pi_{n+k}(\Sigma^n X) \longrightarrow \pi_{n+k+1}(\Sigma^{n+1} X) \longrightarrow \cdots \right)$$

$f \longmapsto f \wedge id$ "$\Sigma(\Sigma^n X) = \Sigma^n X \wedge S^1$"

Example from Wikipedia's Homotopy groups of spheres

From Freudenthal, we get $\pi_{n+k}(S^n) \xrightarrow[\Sigma]{\cong} \pi_{n+k+1}(S^{n+1})$ for $n > k+1$.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	${\mathbb{Z}_2}^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

We see that

$$\pi_k(S^{n+k}) = \pi_{k+1}(S^{n+k+1})$$

for large enough n's.

Is there a nice way to write

$$\pi_k(\,?\,) = \operatorname*{colim}_n \pi_k(S^{n+k})$$

Yeah. It uses Spectra

What are Spectra?

Def$^{\underline{n}}$ a prespectrum T is
- a sequence of spaces$_*$ $\{T_n\}_{n\geq 0}$
- maps$_*$ $\Sigma T_n \longrightarrow T_{n+1}$. Same info $T_n \to \Omega T_{n+1}$

Examples

Suspension Prespectrum

Given X, consider $\{\Sigma^n X\}_{n\geq 0}$ and

$$\Sigma(\Sigma^{n-1}X) \xrightarrow{\cong} \Sigma^n X$$

Sphere prespectrum

$$\mathbb{S}^0 = \{S^0, S^1, S^2, \dots\}$$

$$\Sigma S^n \xrightarrow{\cong} S^{n+1}$$

Def^n A map between prespectra $T \longrightarrow T'$ is

is maps $\{T_n \longrightarrow T_n'\}_{n\geq 0}$ such that

$$\begin{tikzcd} \Sigma T_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & T_{n+1} \arrow[d] \\ \Sigma T_n' \arrow[r] & T_{n+1}' \end{tikzcd}$$

spaces are "de-loopings" of eachother

Def$^{\underline{n}}$ an Ω-prespectrum T is a prespectrum s.t $T_n \xrightarrow{\simeq} \Omega T_{n+1}$ is a homotopy equivalence

$\Sigma T_n \longrightarrow T_{n+1}$

Def^n a spectrum T is a prespectrum s.t $T_n \longrightarrow \Omega T_{n+1}$ is a homeomorphism

Useful Tool: Spectrifying a PreSpectrum.

Given $T = \{T_n\}_{n \geq 0}$ a prespectrum,

Let $LT := \{(LT)_n\}_{n \geq 0}$ where

$$(LT)_n = \bigcup_k \Omega^k T_{n+k}$$

Look! $\Omega (LT)_{n+1} = \Omega\left(\bigcup_k \Omega^k T_{n+k+1} \right)$

$$= \bigcup_k \Omega^{k+1} T_{n+k+1}$$

$= (LT)_n$, so LT is a spectrum!

Now for a based space X, we have the suspension spectrum

$$\Sigma^\infty X = L\{\Sigma^n X\}_{n \geq 0}$$

where $(\Sigma^\infty X)_n = Q \Sigma^n X$

like the 0th space but fully de-looped.

where $QX = \Omega^\infty \Sigma^\infty X = \bigcup_n \Omega^n \Sigma^n X$.

these give us relationships

$$\begin{tikzcd} Top_* \arrow[r, shift left, "\Sigma^\infty"] \arrow[rr, bend left, "\Sigma^\infty"] & PreSp \arrow[l, shift left, "\Omega^\infty"] \arrow[r, shift left, "L"] & Sp \arrow[l, shift left, "U"] \arrow[ll, bend left, "\Omega^\infty"] \end{tikzcd}$$

What about homotopy and homology, Scotty?

Let $\mathbb{T} = \{T_n\}_{n \geq 0}$ be a prespectrum.

Def$^{\underline{n}}$ $\pi_k \mathbb{T} := \operatorname{colim}_n \pi_{n+k} T_n$ Looks a lot like $\pi_k^{st}(X)$

$$= \operatorname{colim}_n \left(\begin{tikzcd} \pi_{n+k} T_n \arrow[rr] \arrow[dr] & & \pi_{n+k+1} T_{n+1} \arrow[r] & \cdots \\ & \pi_{n+k} \Omega T_{n+1} \arrow[ur] & & \end{tikzcd} \right)$$

Def$^{\underline{n}}$ For $\{E_k\}$ a generalized homology theory

$$E_k(\mathbb{T}) = \operatorname{colim}_n \widetilde{E}_{n+k}(T_n)$$

FACT: $\mathbb{T} \longrightarrow L\mathbb{T}$

induces isomorphisms

$\pi_k(\mathbb{T}) \cong \pi_k(L\mathbb{T})$ $\qquad$ $E_k(\mathbb{T}) \cong E_k(L\mathbb{T})$

SO... for spheres, the stable homotopy groups are given by

$$\pi_k^{st} S^0 = \pi_k \mathbb{S}^0 = \pi_{n+k} \mathbb{S}^n$$

$$\mathbb{S}^n = \{\underset{0}{S^n}, \underset{1}{S^{n+1}}, \underset{2}{\cdots}\}$$

Matches up with our stable htpy groups of spheres we noted earlier!!!

More Spectra

- Given an abelian group G, we have the
  Eilenberg MacLane prespectrum $\tilde{H}G = \{K(G,n)\}_n$
  " " spectrum $L(\tilde{H}G)$.

  Note $\Omega K(G,n)$ is a $K(G,n-1)$

- The desuspension of a suspension prespectrum
  $$\Sigma^{-n}\Sigma^{\infty}X = \{\underset{0}{*}, \dots, \underset{n-1}{*}, \underset{n}{X}, \underset{n+1}{\Sigma X}, \underset{n+2}{\Sigma^2 X}, \dots\}$$
  Σ^{-n} feels like a shift

  so for example
  prespectrum $$S^{-n} = \{\underset{0}{*}, \dots, \underset{n-1}{\Omega QS^0}, \underset{n}{QS^0}, \underset{n+1}{QS^1}, \dots\}.$$

Constructing more spectra from old

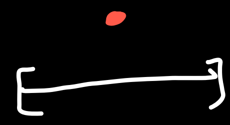
T a prespectrum and X a space

- $T \wedge X = \{T_n \wedge X\}_{n \geq 0}$
- $F(X, T) = \{\mathrm{Maps}(X, T_n)\}_{n \geq 0}$
- $T \vee T' = \{T_n \vee T_n'\}$

$\Sigma T = T \wedge S^1$

$\Sigma T_n \to T_{n+1}$

$\Sigma T_n \wedge X \to T_{n+1} \wedge X$

Let $I_+ = [0,1] \sqcup \{*\}$. 

Def$^{\underline{n}}$ $f, g : T \to T'$ are **homotopic** if

$\exists\ H : T \wedge I_+ \longrightarrow T'$

such that

$$H\big|_{T \vee T} = f \vee g : T \vee T \to T \wedge I_+ \xrightarrow{H} T'$$

Def$^{\underline{n}}$ T is **homotopy equivalent** to T' if $\exists \begin{cases} T \xrightarrow{f} T' \\ T' \xrightarrow{g} T \end{cases}$

s.t $g \circ f \simeq id_T$ and $f \circ g \simeq id_{T'}$

So now we can talk about homotopy classes of maps.

If E is a spectrum, X is a space, G an abelian group.

- $[\mathbb{S}^n, E]_{Sp} = \pi_n E$. = matches with our earlier notion $\operatorname{colim}_n \pi_{n+k}(E_n)$
- $[\mathbb{S}^n, HG \wedge X]_{Sp} = HG_n X = H_n(X;G)$
- $[X, \Sigma^n HG] = HG^n X = H^n(X;G)$.
- $[\Sigma E, \Sigma E']_{Sp} = [E, E']_{Sp} = [\Sigma^{-1} E, \Sigma^{-1} E']_{Sp}$

Σ^{-1} "is" Ω

Takeaways:

- Spectra exist
- We can map between spectra
- they allow us to compute stable homotopy groups
- they allow us to build generalised reduced cohomology theories
- We have plenty of constructions to get new spectra from old.
- We can loop and "de-loop" spectra "feels like shifting"
- negative spheres "make sense"

<u>Questions</u>: What about the homotopy category of Spectra?

~~$\Omega^\infty \Sigma^\infty$uestions?~~

Questions?