Dehn it!

Distinguishing diffeomorphisms with Equivariant Bauer-Furuta Invariants

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November 2, 2023

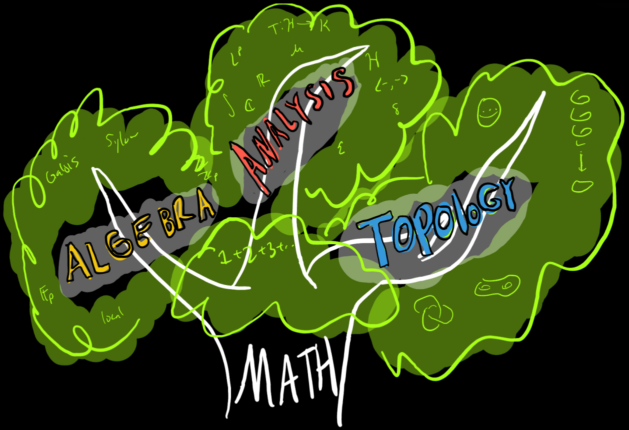
Goals

1. Purpose
    - Share math
    - Ask questions, learn more
    - Introduce you all to what I think about
    - Consolidate the things I think about
2. Successful Outcome
    - You understand where this fits into the mathematical landscape
    - You understand what I'm trying to do, broadstrokes
    - You have an idea of what's been done before
    - At least one person asks me one question
    - I see a part of the presentation I could've made clearer

Plan for today

Context

Where are we visiting



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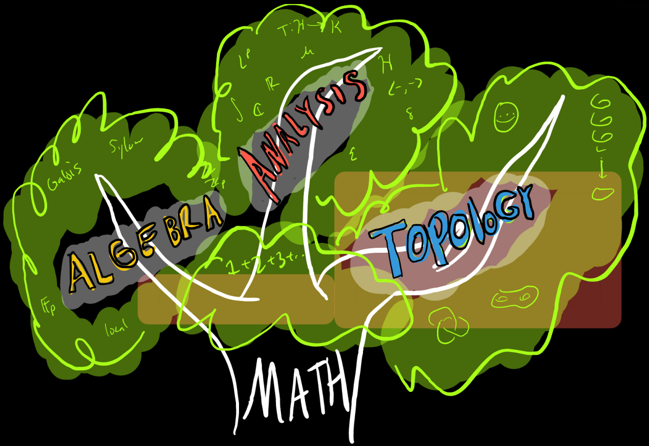

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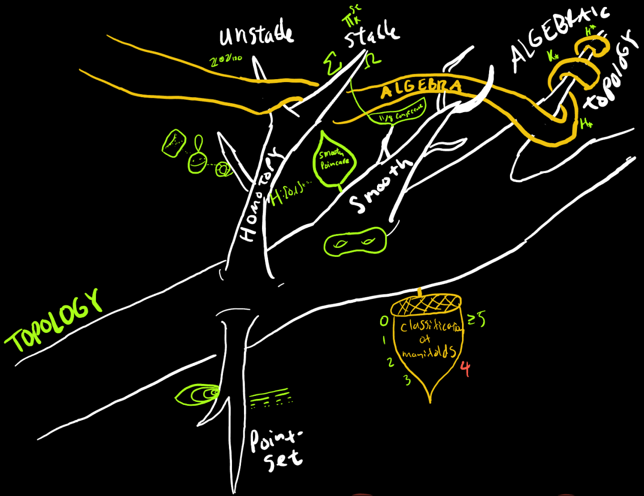

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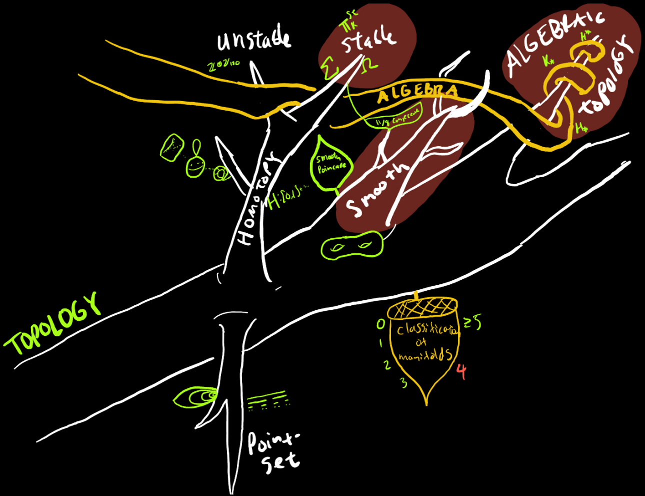

Context

4-manifolds: All our 4-manifolds are simply connected, $\pi_1(X) = 0$.

- (Useful old result) Freedman's classification $(Q, 0 \text{ or } 1^{\text{smooth}}_{\text{not}})$
- (Big topology goal) Smooth Poincare conjecture
  $X \simeq S^4 \overset{?}{\Rightarrow} X \cong_{\text{Diff}} S^4$
- (Big goal) 11/8 conjecture. $X \in \mathsf{Spin4Mfld}^{sm}_{cls}$ iff
  $b_2(X) \overset{?}{\geq} \frac{11}{8}|\sigma(X)|$ or $q \overset{?}{\geq} 3p$ for $2pE_8 \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$
- (Big goal) 4D Smale conjecture $\pi_0(\text{Diff}(S^4)) \overset{?}{\cong} \mathbb{Z}/2$
- Spin-manifolds e.g $S^4, S^2 \times S^2$, K3, non e.g. $\mathbb{CP}^2, \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$

Key things

- (Pin(2)-)Seiberg Witten equations
- Seiberg-Witten invariants
- (Pin(2)-)Bauer-Furuta invariants

Selected History up to 2010

‘81 Matsumoto $\frac{11}{8}$
[Mat81]

‘82 Freedman, $\mathrm{TopMfld}^4$
[Fre82]

‘94 SW Equations
[Wit94]

‘94 Kronheimer $\mathrm{Pin}(2)$ SW
Lecture in 94 at Cambridge

Furuta $\frac{11}{8}$
[Fur01]

‘02 Bauer-Furuta invariant
[BF02]

‘03 Birgit Schmidt’s thesis
[Sch03]

‘04 FBF
[Xu04]

‘08 FBF
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‘15 △ Conjecture
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‘21 Exotic Diffeomorphism using FBF
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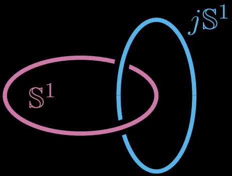
History, more linear

- ∼1981 Matsumoto's 11/8 conjecture [Mat81]
- ∼1982 Freedman classifies 4-manifolds [Fre82]
- ∼1994 Seiberg-Witten equations [Wit94]
- ∼2001 Furuta $10/8|\sigma|+2$ [Fur01]
- ∼2002 Bauer-Furuta invariant [BF02]
- ∼2003 Calculations and work toward 11/8 [Sch03]
- ∼2004 Families Bauer-Furuta [Xu04]
- ∼2008 Families Bauer-Furuta [Szy20]
- ∼2013 Pin(2) intersection[Man13a]
- ∼2015 Disproof of Δ-conj [Man13b]
- ∼2018 Work towards 11/8 [HLSX18]
- ∼2020 Exotic Diffeo [KM20]
- ∼2021 Exotic Diffeo [Lin20]
- ∼2023 Exotic Diffeo 2023 Exotic Diffeo on seifert fibered manifold[KMT23]

Definitions

- $\mathsf{Pin}(2)$
- $\mathsf{Spin}(n), \mathsf{Spin}^c(n)$
- Seiberg-Witten Equations
- equivariant stable homotopy theory
- Bauer-Furuta invariant
- Families Bauer-Furuta invariant
- Dehn twist

Definitions:

- $\mathsf{Pin}(2)$ 

$\subset \mathbb{C} \oplus j\mathbb{C} = \mathbb{H} = \mathsf{Cl}(\mathbb{R}^2)$,

  or, also a particular double cover of $\mathsf{O}(2)$
- $\mathsf{Spin}(n)$ = double cover of $\mathsf{SO}(n)$,or also, $\mathsf{Pin}(n) \cap \mathsf{Cl}_0(\mathbb{R}^n)$

  Examples:

n	1	2	3	4
$\mathsf{Spin}(n) \cong_{top}$	S^0	S^1	S^3	$S^3 \times S^3$

- $\mathsf{Spin}^c(n) \cong \mathsf{Spin}(n) \times_{\{\pm 1\}} S^1$,
  or, also the double covering of $\mathsf{SO}(n) \times S^1$ which is nontrivial on both factors.

Spin structure

Let X be an oriented 4-manifold, so there is a principal bundle

$$\begin{array}{ccc} \mathsf{SO}(4) & \hookrightarrow & \mathrm{Fr}(TX) \\ & & \downarrow \\ & & X \end{array}$$

A Spin structure is a bundle P such that the following commutes

$$\begin{array}{ccc} \mathsf{Spin}(4) & \longrightarrow & P \\ \downarrow{\scriptstyle \times 2} & & \downarrow \\ \mathsf{SO}(4) & \hookrightarrow & \mathrm{Fr}(TX) \\ & & \downarrow \\ & & X \end{array}$$

There is a way to make a Spin^c-structure. See [Mor96]

Spin structure

FACT:

A vector bundle $E \to X$ admits a Spin structure iff $w_2(E) = 0$.

A principal spin structure gives us

$$\begin{array}{ccccc} & & S^+ & & \\ & \nearrow & & \searrow & \\ P_{\mathsf{Spin}(4)} = P_{SU(2)\times SU(2)} & & & & X \\ & \searrow & & \nearrow & \\ & & S^- & & \end{array}$$

such that $S^+ \otimes_{\mathbb{H}} (S^-)^* \cong TX$.

Watch Stipsicz video from minicourse for more.

Spin structure

Fact

When spin structures exist on M (any manifold),

$$\{\text{Spin-structures}\} \overset{1:1}{\longleftrightarrow} H^1(M;\mathbb{Z}/2).$$

Given a fiber bundle $\begin{matrix} X & \hookrightarrow & E \\ & & \downarrow\pi \\ & & B \end{matrix}$, you can take the vertical tangent bundle $T^vE := \ker(D\pi)$, to get a vertical frame bundle

$$\begin{matrix} \mathrm{SO}(4) & \hookrightarrow & \mathrm{Fr}^v(T^vE) \\ & & \downarrow \\ & & B \end{matrix}.$$

When $w_2(T^vE) = 0$, you get a Spin structure.

Seiberg-Witten equations v.1

Seiberg Witten Equations

A set of nonlinear elliptic differential equations.

Their solutions satisfy a nice compactness condition that you don't run into all the time.

Seiberg-Witten equations v.2

Let A be a unitary connection on $L := \det(S^{\pm}(P))$
Let $\partial\!\!\!/_A$ be the Dirac operator associated to $\nabla_{L.C}$ on $\mathsf{Fr}(TX)$ and A.
Let $\psi \in C^{\infty}(S^{+}(P))$
Let h be a generic, self-dual two form on X

Seiberg-Witten Equations [Mor96, 4.2s]

$$\begin{aligned} L^2_2(\overbrace{(T^*X \otimes i\mathbb{R}) \oplus S^+(P)}^{T_p\mathcal{C}(P)}) &\xrightarrow{F} L^2_1((\Lambda^2_+ T^*X \otimes i\mathbb{R}) \oplus S^-(P)) \\ (A, \psi) &\longmapsto (F^+_A - q(\psi) - ih, \partial\!\!\!/_A(\psi)) \end{aligned}$$

Equation: $F(A, \psi) = 0$

Seiberg-Witten Equations v.2

Seiberg-Witten Equations [Mor96, 4.2s]

$$\overbrace{L^2_2((T^*X \otimes i\mathbb{R}) \oplus S^+(P))}^{T_p\mathcal{C}(P)} \xrightarrow{F} L^2_1((\Lambda^2_+ T^*X \otimes i\mathbb{R}) \oplus S^-(P))$$

$$(A, \psi) \longmapsto (F_A^+ - q(\psi) - ih, \partial\!\!\!/_A(\psi))$$

$$\textbf{Equation: } F(A, \psi) = 0$$

Theorem

Suppose that $b_2^+(X) > 1$.
$\mathcal{M}_P := F^{-1}(0)_{/\mathsf{Maps}(X,S^1)}$ *is a closed, orientable manifold of dimension*

$$\frac{c_1(L)^2 - 2\chi(X) - 3\sigma(X)}{4}.$$

Seiberg witten invariant

Suppose $\dim(\mathcal{M}_P) = d$ is even.
Let μ be a specific Chern class with an association to P defined in [Mor96].Then

$$SW(X, -) : \overbrace{\begin{Bmatrix} \text{space of} \\ \text{Spin}^c \text{ strs} \end{Bmatrix}}^{\cong_{\text{not natural}} H^2(X;\mathbb{Z})} \longrightarrow \mathbb{Z}$$

$$P \longmapsto \int_{\mathcal{M}_P} \mu^{d/2}$$

Important case ($d = 0$): $\displaystyle\int_{M_P} \mu^0 = \sum_{p \in \mathcal{M}_P} \text{sign}(p)$

($d > 0$?)

Equivariant Things

- We're in an incomplete universe $\mathcal{U}$ with countably many irreducible representations of $\mathsf{Pin}(2)$ that matter to us.
- Most important representations today

$$\text{trivial: } \mathbb{R}, \quad \text{j-flip: } \widetilde{\mathbb{R}}, \quad \text{quaternionic: } \mathbb{H}$$

- $nV := V^{\oplus n}$,
- $S(nV) := \{v \in nV \mid \|v\| = 1\}$,
- $S^{nV} := (nV)^+$
- $M_+ := M \sqcup \{*\}$
- $\{X, Y\}^{\mathsf{Pin}(2)}$ is the $\{$stable, pointed $\mathsf{Pin}(2)$-equiv. maps $/ \simeq\}$
- $\{X, Y\}^{\mathsf{Pin}(2)} := \lim\limits_{\substack{V \subset \mathcal{U} \\ f.d}} [S^V \wedge X, S^V \wedge Y]^{\mathsf{Pin}(2)}$

Bauer-Furuta invariant

Nonequivariant version

$\tilde{\mathcal{M}}_P := F^{-1}(0)_{/\mathsf{Maps}_*(X,S^1)}$ is a stably framed manifold (submanifold of a Hilbert space), and so is represented in a stable homotopy group of spheres, and so $BF(X,P) = [\tilde{\mathcal{M}}_P] \in \pi^{st.}_{d+1}$

Equivariant versions

Let $n = \frac{-\sigma(X)}{16}$ and $m = b_2^+(X)$

- $BF^{S^1}(X) \in \left\{S^{2n\mathbb{C}}, S^{m\mathbb{R}}\right\}^{S^1}$,and
- $BF^{\mathsf{Pin}(2)}(X) \in \left\{S^{n\mathbb{H}}, S^{m\tilde{\mathbb{R}}}\right\}^{\mathsf{Pin}(2)}$

Pros and Cons of each BF

Some intuition

- $BF(X)$- likely stronger than SW-invariant
- $BF^{S^1}(X)$- recovers SW-invariant, more symmetry
- $BF^{\mathsf{Pin}(2)}(X)$- More structure, more refined, more symmetry

S^1-Bauer-Furuta → Seiberg-Witten invariant

We'll use BF^{S^1} here.

Recovery of SW-invariant: Let $m-1 \leq 2n-1$.

Suppose we have a map $S^{2n\mathbb{C}} \xrightarrow{F_{\text{approx}}} S^{m\mathbb{R}}$ Then we have a cofiber sequence

$$S^0 \to S^{2n\mathbb{C}} \to \Sigma S(2n\mathbb{C})_+ \to S^1$$

$$\underbrace{\{S^1, S^{m\mathbb{R}}\}^{S^1}}_{0} \to \underbrace{\{\Sigma S(2n\mathbb{C})_+, S^{m\mathbb{R}}\}^{S^1}}_{\{S(2n\mathbb{C})_+, S^{(m-1)\mathbb{R}}\}^{S^1} = \{\mathbb{CP}_+^{2n-1}, S^{m\mathbb{R}}\}} \xrightarrow{\cong} \{S^{2n\mathbb{C}}, S^{m\mathbb{R}}\}^{S^1} \to \underbrace{\{S^0, S^{m\mathbb{R}}\}^{S^1}}_{0}$$

$$\left(S^{2n\mathbb{C}} \xrightarrow{F_{\text{approx}}} S^{m\mathbb{R}}\right) \longmapsto \left(\mathbb{CP}_+^{2n-1} \xrightarrow{\tilde{f}} S^{(m-1)\mathbb{R}}\right) \longmapsto \pi^{m-1}(\mathbb{CP}^{2n-1})$$

$$\pi^{m-1}(\mathbb{CP}^{2n-1}) \xrightarrow{\tilde{f}(1)} H^{m-1}(\mathbb{CP}^{2n-1}) = \begin{cases} \mathbb{Z} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

An instance of equivariant mapping degree!

Families Bauer-Furuta invariant

- $$\begin{array}{ccc} X & \hookrightarrow & X \\ & & \downarrow \\ & & * \end{array}$$ gives a bundle map $$\begin{array}{ccc} S^{4n\mathbb{R}+C} & \xrightarrow{F} & S^{m\mathbb{R}+C} \\ & \searrow \quad * \quad \swarrow \!\! \nearrow_{s_0} & \end{array}$$

  $$BF(X) = F^{-1}(s_0(*)) \in \pi^{st}_{d+1+\dim(*)}$$

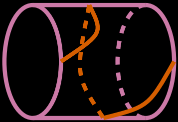
- $$\begin{array}{ccc} X & \hookrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$ gives a bundle map $$\begin{array}{ccc} \tilde{S}^{4n\mathbb{R}+C} & \xrightarrow{F} & \tilde{S}^{m'\widetilde{\mathbb{R}}+C} \\ & \searrow \quad B \quad \swarrow \!\! \nearrow_{s_0} & \end{array}$$

  $$FBF(E,\mathfrak{s}) = F^{-1}(s_0(B)) \in \pi^{st}_{d+1+\dim(B)}$$

Dehn twist

- $\pi_1(\mathsf{SO}(4)) = \mathbb{Z}/2$
- take a nontrivial loop $[\gamma] \in \pi_1(\mathsf{SO}(4))$.
- Then you have a map

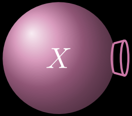
$$S^3 \times [0,1] \longrightarrow S^3 \times [0,1]$$
$$(x,t) \longmapsto (\gamma(t)\cdot x, t)$$

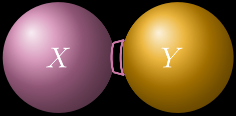


Key examples of $\delta \in \mathrm{Diff}_\partial(\mathring{X}, \partial\mathring{X})$ or $\delta \in \mathrm{Diff}(X)$:

$\mathring{X} := X \setminus B^4$

$X \# Y$





Key Results in our world today

- Kronheimer-Mrowka
- Jianfeng Lin

Kronheimer-Mrowka

Theorem ([KM20])

The dehn twist on K3#K3 is not smoothly isotopic to the identity in Diff(K3#K3)

Key ideas

- Mapping torus M_δ of Dehn twist along K3#K3 (2 spin structures since $H^1(M_\delta; \mathbb{Z}/2) = \mathbb{Z}/2$).
- Use BF on M_δ
- $BF(\text{K3}) = \eta$
- Lemma on $X\#_\alpha Y$ giving a product formula for Families Bauer furuta
- $0 \neq \eta^3 \in \pi_3^{st} = \mathbb{Z}/24$
- The dehn twist can't be isotopic to 1

Picture Frame if we'd like

Jianfeng Lin

Theorem ([Lin20])

The dehn twist on K3#K3#$(S^2 \times S^2)$ is not smoothly isotopic to the identity in Diff(K3#K3#$(S^2 \times S^2)$).
First instance of exotic structure staying after one stabilization, i.e #$S^2 \times S^2$

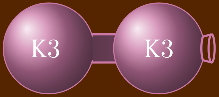
Key Ideas

- Create a mapping torus along the dehn twist and compare the spin structures.
- Use $BF^{\mathsf{Pin}(2)}$
- Reduce to nonequivariant.
- Find a contradiction $0 \neq \eta^3$ from [KM20].

More Dehn twists to look at

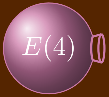
Question 1

Is the dehn twist isotopic to the identity on $(K3\#K3)^\circ$?



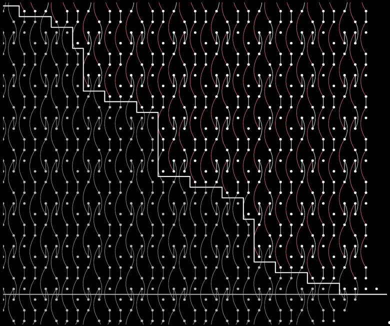
Question 2

Is the dehn twist isotopic to the identity on $(E(4))^\circ$?



Potential Ideas

Potential ideas: Follow some ideas from [HLSX18] or work out some equivariant K-theory computations.

,

$$R(\mathsf{Pin}(2)) \cong \frac{\mathbb{Z}[z,w]}{(zw = 2w = w^2)}$$

$$RO(\mathsf{Pin}(2)) \cong \frac{\mathbb{Z}[D,K,H]}{\text{lots-o-relns}}$$

The End!

Thank You!

Questions?

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