$X \to E$
$\downarrow$
B

UCSD arXiv Seminar
2310.18695

Smooth
Continuous

$F_A^+ = \sigma(\Phi\Phi) + \mu$

$D_A \Phi = 0$

More evidence for the misbehavior of smooth 4-manifolds

$U(1)$

Scotty Tilton
UCSD
February 6, 2024

$BDiff(X)$

$c_i \in H^*(\mathbb{C}P^\infty; \mathbb{Z})$

$$\begin{array}{ccc} S^3 & & (z,w) \\ \downarrow & & \downarrow \\ S^2 & & w/z^m \end{array}$$

$\Omega_g^+(X)$

Plan 4 Today

1. Context and Suspense

2. Big Theorem

3. Techniques Used for Big Theorem
   - cute facts about BG I like.
   - characteristic classes
   - Seiberg-Witten theory
   - Some neat constructions + pf

4. Future Directions

Some Facts and Context

Sullivan '77 Let $n \geq 5$. If X is a simply connected closed n-manifold, then $\underbrace{\pi_0(\mathrm{Diff}(X)) \text{ is finitely generated}}_{\star}$.

Let $n \leq 3$. If X is a closed n-manifold. Then $\pi_0(\mathrm{Diff}(X))$ is finitely presented.

(See Dehn in a Hatcher-Thurston paper and Hong-McCullough)

What about dimension 4?

We'll focus our attention on X with $\pi_1(X) = 0$.

Nice places to look for answers

- $X = S^4$? Any info on $\pi_0(\mathrm{Diff}(S^4))$ gets you a Fields medal, I reckon.

  Pretty tough!

- $X =$ another 4 manifold?

Contenders: $S^2 \times S^2$, $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, $n\mathbb{C}P^2 \# m\overline{\mathbb{C}P^2}$, $K3$,

$$E(k) \begin{cases} E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2} \\ E(2) = K3 \\ E(3) \cdots \text{ etc.} \end{cases},$$

Any info we know?

1

- Quinn '86, Perron '86 $\pi_0(\mathrm{Homeo}(X))$ is finitely generated.

$n \geq 4$

- If $\pi_1(M^n) \neq 0$, then $\pi_0(\mathrm{Diff}(M))$ could or could not be ∞-generated.

- Bustamante – Krannich – Kupers '23 showed for $M^{\text{even} \geq 6}$, $|\pi_1(M)| < \infty$, $H_*(B\mathrm{Diff}(M))$ is finitely generated.

see
- Hatcher '78 showed $\pi_0(\mathrm{Diff}(T^k))$ inf gen for $k \geq 5$.
- Budney - Gabai 2021, Watanabe 2023 for dim 4.
- Budney - Gabai show $\pi_0(\mathrm{Homeo})$ is even bad.

Big Theorem (Konno '23)

$\pi_0(\mathrm{Diff}(E(n)\#S^2\times S^2))$ isn't finitely generated

$\Uparrow$

General Result (Konno '23) Let X be a simply connected, closed 4-manifold satisfying some technical assumptions (E(n) is such an example for $n\geq 1$). Then for $k>0$,

$$\mathrm{Ker}\left(H_k(B\mathrm{Diff}^+(X\#kS^2\times S^2)) \xrightarrow{i_*} H_k(B\mathrm{Homeo}^+(X\#kS^2\times S^2))\right)$$

contains a subgroup isomorphic to

$$(\mathbb{Z}/2)^{\oplus\infty}.$$

Techniques: Cute BG facts

③

$$B : \text{"TopGrps} \longrightarrow \text{Top"}$$
$$G \longmapsto BG$$

$$\begin{array}{c} G \hookrightarrow EG \simeq * \\ \downarrow \\ BG \end{array}$$ is a fibration, inducing a l.e.s on π_k:

$$\cdots \to \underbrace{\pi_2(EG)}_{0} \to \underbrace{\pi_2(BG) \to \pi_1(G)}_{\cong} \to \underbrace{\pi_1(EG)}_{0} \to \underbrace{\pi_1(BG) \to \pi_0(G)}_{\cong} \to \pi_0(EG)$$

i.e $\pi_{n+1}(BG) \cong \pi_n(G)$

Homotopy classes of maps to BG classify principal G-bundles.

Why? • EG has a right G-action

$$EG = \bigcup_{\vec{g} \in G^\infty} \underset{\Delta^n}{[g_0, \dots, g_n]}$$ where faces are glued in a very straightforward way

•
$$\begin{tikzcd} G \arrow[r, hook] & EG \arrow[d] \\ & BG \end{tikzcd} \Rightarrow \begin{tikzcd} f^*EG \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$ gives a G bundle.

You can make a vector bundle by taking a G-rep V and doing
$$\begin{tikzcd} V \arrow[r, hook] & f^*EG \times_G V \arrow[d] \\ & X \end{tikzcd}.$$

Ex

- $\mathbb{R}P^\infty$ is a $B\mathbb{Z}/2$ — $E = S^\infty$
- S^1 is a $B\mathbb{Z}$ — $E = S^\infty$
- $\mathbb{C}P^\infty$ is a BS^1 — $E = S^\infty$
- $Gr_k(\mathbb{R}^\infty)$ is a $BO(k)$ — $E = V_k(\mathbb{R}^\infty)$
- $Gr_k(\mathbb{C}^\infty)$ is a $BU(k)$ — $E = V_k(\mathbb{C}^\infty)$

$B\mathrm{Diff}(X)$ classifies smooth X-bundles

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E\mathrm{Diff}(X) \arrow[d] \\ B \arrow[r] & B\mathrm{Diff}(X) \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathrm{Diff}(X) \arrow[r, hook] & E \arrow[d] \\ & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} X \arrow[r, hook] & T = E \times_{\mathrm{Diff}(X)} X \arrow[d] \\ & B \end{tikzcd}$$

$B\mathrm{Homeo}(X)$ classifies topological X-bundles.

We get characteristic classes from

$H^*(BG;\mathbb{Z})$ by

$$B \longrightarrow BG \rightsquigarrow H^*(BG) \longrightarrow H^*(B).$$

Ex

- Stiefel-whitney classes $H^*(BO(n);\mathbb{Z}/2) = \mathbb{Z}/2[w_1, \dots, w_n]$
  $|w_i| = i$
- Chern-classes $H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_n]$
  $|c_i| = 2i$
- euler classes $H^*(BSO(n);\mathbb{Z})$
- Us today $\mathbb{SW}^{k}_{i,\Sigma^{i+1}}(X,\xi) \in H^k(BDiff^+(X);\mathbb{Z}/2)$

Seiberg Witten theory

X a simply ctd 4-mfld w/ a Spin^c structure. $\mathfrak{s}$

Note: there are $H^2(X;\mathbb{Z})$ spin^c structures on X.

then the differential equations

$$F_A^+ = \sigma(\Phi, \Phi) + \mu$$
$$D_A \Phi = 0$$

modulo gauge action
has a compact solution space of dimension

$$d(\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)}{4}.$$

Let $\mathrm{Spin}^c(X,k) = \left\{ \begin{array}{c} \text{spin}^c \text{ str's on } X \text{ w/} \\ d(s) = -k \end{array} \right\}$.

$\mathbb{Z}/2 \curvearrowright \mathrm{Spin}^c(X,k)$ $\quad c_1(-1\cdot s) = -c_1(s)$.

$\mathrm{Diff}^+(X) \curvearrowright \mathrm{Spin}^c(X,k)$, $\quad s \longmapsto \varphi^* s$.

the $\mathrm{Diff}^+(X)$ action commutes w/ $\mathbb{Z}/2$ action.

Let $\mathfrak{S} \subset \mathrm{Spin}^c(X,k)/_{\mathbb{Z}/2}$ s.t.

$$\mathrm{Diff}^+(X)\cdot \mathfrak{S} = \mathfrak{S}.$$

detailed construction + generalization of Lin-Konno + see paper for more details!

the cohomology class used is

$$\mathbb{SW}^k_{\frac{1}{2}\text{-tor}}(X, \mathfrak{S}) \in H^k(B\mathrm{Diff}^+(X); \mathbb{Z}/2).$$

General Result (Konno '23): Let X be a simply connected, closed 4-manifold satisfying some technical assumptions (E(n) is such an example for $n \geq 1$). Then for $k > 0$,

$$\operatorname{Ker}\left(H_k\left(B\mathrm{Diff}^+(X \# kS^2 \times S^2)\right) \xrightarrow{i_*} H_k\left(B\mathrm{Homeo}^+(X \# kS^2 \times S^2)\right)\right)$$

contains a subgroup isomorphic to

$$(\mathbb{Z}/2)^{\oplus\infty}.$$

④

Proof of theorem

Set $M = E(n)$, and for $i \geq 1$ $M_i = E(n,i)$.

SW-invariants done by Fintushel-Stern 1997

Fact

$$E(n) \underset{\text{Homeo}}{\cong} E(n,i)$$

$$E(n) \# S^2 \times S^2 \underset{\text{Diffeo}}{\cong} E(n,i) \# S^2 \times S^2 .$$

Gompf '91

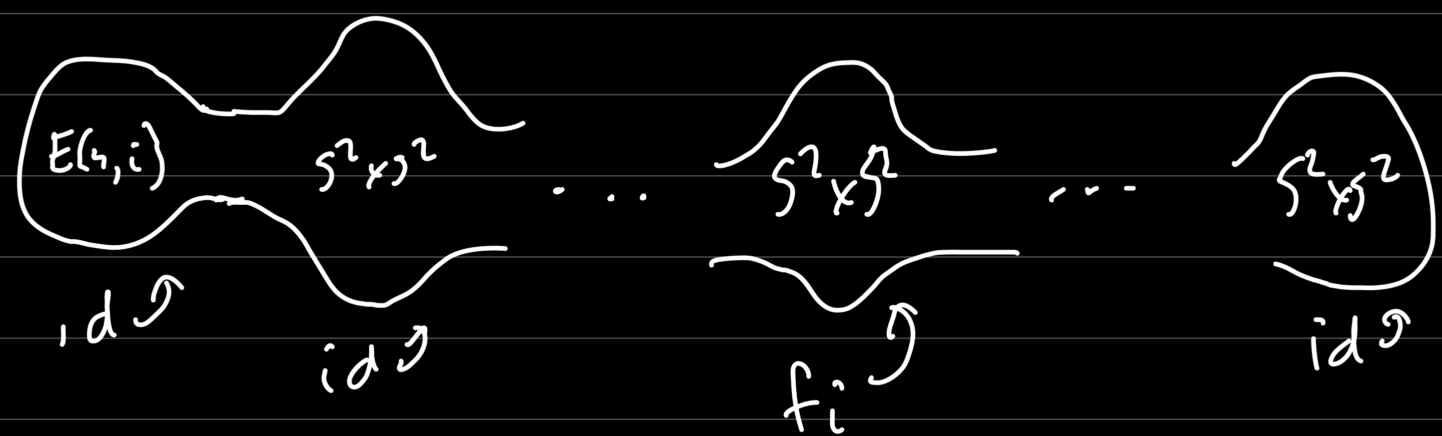
Multiple mapping tori

Let $f_0 : S^2 \times S^2 \longrightarrow S^2 \times S^2$
$(x, y) \longmapsto (\mathrm{refl}_z x, \mathrm{refl}_z y)$

Let $f \simeq f_0$ where $f|_{D^4} \equiv id$ for some $D^4 \in S^2 \times S^2$.

Take copies $\tilde{f}_1, \dots \tilde{f}_n$ and make diffeos

$f_j : E(n, i) \# k S^2 \times S^2$



Note f_j commute since the supports are $\perp$.

Make a multiple mapping torus

$$E(n,i)\#kS^2\times S^2 \hookrightarrow E_i \downarrow T^k.$$

For each i, we have (by abuse of notation)

$$E_i : T^k \longrightarrow BDiff^+(\underbrace{E(n)\#kS^2\times S^2}_{X}).$$

Let $\alpha_i := (E_1)_*([T^k]) - (E_i)_*([T^k]) \in H_k(BDiff^+(X))$

Lemma For $i: BDiff^+(X) \longrightarrow BHomeo^+(X)$,

$$i_* \alpha_i = 0.$$

pf $E(n) \underset{\text{homeo}}{\cong} E(n,i) \Rightarrow E_i \underset{\text{homeo}}{\cong} E_1$

$$\Rightarrow \quad i_*(E_1)_*([T^k]) - i_*(E_i)_*([T_n])$$

$$\parallel$$

$$(E_1^{top})_*([T^k]) - (E_i^{top})([T_n]) = 0. \quad \square.$$

Technical part | α_i are linearly independence.

Showing

$$\bigoplus_{i\geq 2}\langle SW^k_{tot}(X,\mathfrak{s}_i), -\rangle := \varphi : BDiff^+(X;\mathbb{Z}) \to \bigoplus_{i\geq 2}\mathbb{Z}/2$$

that $\{\varphi(\alpha_i)\}_{i=2}^{\infty}$ is linearly independence.

$$\Longrightarrow (\mathbb{Z}/2)^{\oplus\infty} \cong \mathrm{span}_{\mathbb{Z}/2}\{\alpha_i\}_2^{\infty} \subset \ker i_* \subset H_k(BDiff^+(X)).$$

so for $k=1$ we have

$$(\mathbb{Z}/2)^{\oplus\infty} \subset H_1(BDiff^+(X)) = \pi_1(BDiff^+(X))_{ab} = \pi_0(Diff^+(X))_{ab}.$$

Further Directions

① Is $H_k(B\mathrm{Homeo}(X^4))$ necessarily finitely generated?

$$B\mathrm{Homeo}(X) \qquad B^{\text{"formally smooth"}} \qquad B\mathrm{Diffeo}(X)$$

② Is $H_k\left(B^{\text{"formally smooth"}}(X^4)\right)$ finitely generated?

see Lin, Xie '23

Thank
you!