$\mathbb{C}$

$\cdots \to X_1 \to X_0 \to X_{-1} \to \cdots$

$\pi_{*,*}$

Not UFC
Not UCT
Not tmf
Not MMA

$\Gamma_*$

MU

$Sp^{\mathbb{Z}^{op}}$

# mmf

($\mathbb{C}$-motivic modular forms)

$E_2^{s,t,w}(X)$

~~$\mathbb{A}^1$-0~~

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Sp

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## Plan for today:

**Purpose:**

- Overview of Gheorghe-Isaksen-Krause-Ricka
  arXiv: 1810.11050
- Learn something
- Add characters to our story of motivic homotopy
- Not go too into detail, give sketches and ideas. Sometimes very loose.

**Successful Outcome:**

- You can tell me why this seems cool (or at least interesting/an endeavor that makes sense)
- I get at least 1 question
- No one leaves the talk if I joke about MMA

## Organization:

Introduction: • Origin story of our mmf fighter

Rising action: • def-o-categories, functors, $A$, oh my!
• tmf (not TMZ) shows up

Climax: • homotopy type of $\Gamma_\star$ modules?

Falling Action: directions the authors point mmf to

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## Origin Story

We love $\pi_*^{st} S^0$.

the spectrum tmf carries a lot of info about $\pi_*^{st} S^0$.

- elements of $\pi_*^{st} S^0$ which come from tmf are well-understood
  elements of $\pi_*^{st} S^0$ which don't come from tmf are not well-understood
- differentials of ASS/ANSS which come from tmf are well-understood
  differentials of ASS/ANSS which don't come from tmf are not well-understood

# Origin Story

working over $\mathbb{C}$

$\pi_*^{st} S^0$

We love $\pi_{*,*}^{st,mot} S^{0,0}$.

motivic cohomology $\begin{cases} H^* \\ K^* \\ \text{Cobordism} \end{cases}$

Is there an motivic modular forms spectrum which gives a lot of info about $\pi_{*,*}^{st,mot} S^{0,0}$?

First guess: Same approach as tmf.

Requirements: Understand - motivic elliptic cohomology

- moduli spaces of $E_\infty$ structures

Any other guesses?

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Can we keep it in Topology-Land?

Answer: $\mathbb{C}$-motivic modular forms by GIKR.

No smooth schemes, No affine lines, No $\mathbb{A}^1 - 0$

But How??

Wait and see...

## Rising action Categories, Functors, Algebras oh my!

$Sp$ is the category of 2-complete spectra.

$Sp^{\mathbb{Z}^{op}}$ is the category of filtered spectra

objects: $X_* = \cdots \to X_1 \to X_0 \to X_{-1} \to \cdots$

morphisms: $X_* \to Y_* = \{X_s \to Y_s \mid s \in \mathbb{Z}^{op}\}$ that are homotopically coherent

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## Facts about $Sp^{\mathbb{Z}^{op}}$

weak equivalences: $f : X_* \xrightarrow{w.e} Y_*$ if $X_s \xrightarrow{w.e} Y_s \ \forall s$.

symmetric monoidal: $(X_* \otimes Y_*)_s := \operatorname{hocolim}_{i+j \geq s} X_i \wedge Y_j$

Define $S^{s,w} := \cdots \to \underset{w+2}{*} \to \underset{w+1}{*} \to \underset{w}{S^s} \to \underset{w-1}{S^s} \to \cdots$

suspensions: $S^{s,w} \otimes (-) : Sp^{\mathbb{Z}^{op}} \longrightarrow Sp^{\mathbb{Z}^{op}}$

$X_* \longmapsto S^{s,w} \otimes X_* =: \Sigma^{s,w} X_*$

Define $\pi_{s,w} X_* = [S^{s,w}, X_*]$

**Lemma:** $Sp^{\mathbb{Z}^{op}}$ is generated by $S^{0,w}$.

## t-structure on $Sp^{\mathbb{Z}^{op}}$

**Prop:** The following gives $Sp^{\mathbb{Z}^{op}}$ a t-structure.

$Sp^{\mathbb{Z}^{op}}_{\geq 0}$ consists of $X_*$ s.t $\pi_{s,w} X_* = 0$ for $s < 2w$
i.e. $\pi_s X_w = 0$

$Sp^{\mathbb{Z}^{op}}_{\leq 0}$ consists of $X_*$ s.t $\pi_{s,w} X_* = 0$ for $s > 2w$

**Lem + Cor:** $Sp^{\mathbb{Z}^{op}}$ is closed under Day convolution and hence $(\tau_{\geq 0})_*$ is lax symmetric monoidal.

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## Functors!

Let $MU^{\bullet+1} := (\Delta \xrightarrow{MU^{\bullet+1}} Sp)$ s.t $n^{th}$ term is $MU^{\wedge n+1}$ and faces/degeneracies induced by mult on $MU$.

(used for MU-based ASS)

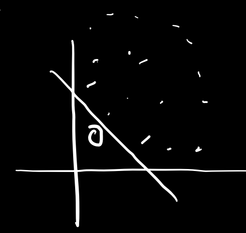
**Def-o-$\Gamma_*$**

$$Sp \xrightarrow{\Gamma_*} Sp^{\mathbb{Z}^{op}}$$

$$X \longmapsto \Gamma_* X = (\cdots \to \underset{w}{\mathrm{Tot}\, \tau_{\geq 2w}}(X \wedge MU^{\bullet+1}) \to \cdots)$$

**Prop on $\Gamma_\star$**: $X \in Sp$ $\quad w \in \mathbb{Z}^{op}$. $\exists$ a s.s.

$$E_2^{s,f,w}(X) = \begin{cases} \mathrm{Ext}^{s,f}_{MU_*MU}(MU_*, MU_*X) & s+f \geq 2w \\ 0 & \text{else} \end{cases}$$



s.t.

$$E_2^{s,f,w}(X) \Longrightarrow \pi_{s,w}\Gamma_\star X$$

compatible w/ the filtered spectrum structure.

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## Algebra

**Theorem** $\quad \Gamma_\star S^0$ is an $E_\infty$-ring object in $Sp^{\mathbb{Z}^{op}}$.

**Prop** $\quad \Gamma_\star$ is lax symmetric monoidal

**pf]** Composition of lax symmetric monoidal functors

**pf]** lax symmetric monoidal functors preserve $E_\infty$-ring objects.

**properties** • $\Gamma_\star$ is sometimes exact

• $\mathrm{Mod}_{\Gamma_\star S^0}$ is generated under hocolim by by $\Sigma^{s,w}\Gamma_\star S^0$

• $\Gamma_\star(\Sigma^{2k}X)$ w.e $\Sigma^{2k,k}\Gamma_\star X$

Prof • if $X = \mathrm{colim}(* \to X^{(1)} \to X^{(2)} \to \cdots)$ with cofibsequences $X^{(n)} \to X^{(n+1)} \to S^{2k_n}$ and $k_n \xrightarrow{n\to\infty} \infty$, and $Y$ similar,

then $\Gamma_\star(X) \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star Y \longrightarrow \Gamma_\star(X\wedge Y)$ is an equivalence.

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## More Algebra:

Let $A_{*,\star} := \pi_{*,\star}\left(\Gamma_\star H\mathbb{F}_2 \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star H\mathbb{F}_2\right)_{*,\star}$

**Theorem** $A_{*,\star} \cong \dfrac{\mathbb{F}_2[\tau][\tau_0,\tau_1,\ldots,\xi_1,\xi_2,\ldots]}{\tau_i^2+\tau\xi_{i+1}}$

$\deg(\tau) = (0,-1)$

$\deg(\xi_i) = (2^{i+1}-2, 2^i-1)$

$\deg(\tau_i) = (2^{i+1}-2, 2^i-1)$

w/ comultiplication.

**pf** They compute $H_{*,\star}(\Gamma_\star BP\langle n\rangle)$ and plug in 0.

## Recap so far

- We have a nice functor $\Gamma_{\star} : Sp \to Sp^{\mathcal{I}^{op}}$ which gives nice homotopical properties like $\Gamma_{\star} S^0$ is an $E_\infty$-ring object and we have t-structures.
- We also can compute the Steenrod algebra which is nice to have.
- Let's see what happens when we plug tmf in.

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# Main $\Gamma_{\star}$ tmf result

**Theorem** $H^{*,\star}(\Gamma_{\star}\, tmf) \cong A/\!/A(2)$

where $A$ is the dual of $A_{*,\star}$ and

$A(n)$ is the dual of $A(n)_{*,\star} := A/_{\xi_1^{2n}, \xi_2^{2^n-2}, \ldots, \xi_n^2, \tau_n^2}$.

**pf idea**
- Make even cell complexes to use results from earlier
- use a relationship of [illegible]
  Cofiber $w_1$ with tmf and $BP\langle 2\rangle$
- calculate calculate calculate.

Takeaway of last result

$H^{*\star}(\Gamma_\star \mathrm{tmf}) = A/\!/A(2)$ is <u>Very</u> similar to the fact that in 2-complete spectra

$$H^*(\mathrm{tmf}) = A/\!/A(2).$$

So $\Gamma_\star \mathrm{tmf}$ is like our <u>mmf!</u>

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Comparison to $\mathbb{C}$-motivic homotopy theory

$\boxed{\text{Theorem}}$ $Sp_{\mathbb{C}}$ is equivalent to $\mathrm{Mod}_{\Gamma_\star S^0}$

tiny proof sketch

construct $\Omega_s^{0,\star} : Sp_{\mathbb{C}} \longrightarrow \mathrm{Mod}_{\Gamma_\star S^0}$

where $\underset{\text{"spectrum"}}{F_s}(X, Y)$ = regular mapping spectrum from motivic $X, Y$

and $\Omega_s^{0,\star}(X) := (\cdots \to F_s(S^{0,n+1}, X) \to F_s(S^{0,n}, X) \to \cdots)$

and it's adjoint, then show they are inverse equivalences. $- \otimes S^{0,n}$

## Takeaways!

- Can do $\mathbb{C}$-motivic problems in $\Gamma_\star S^0$-module world!
- Isaksen + IWX used $\mathbb{C}$-motivic to compute stable homotopy groups of spheres.
- This work doesn't address the motivic world over a field of nonzero characteristic.
- Maybe there is a topological basis also for $\mathbb{R}$-motivic spectra? Burklund-Singer "Galois..."

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# Thank You!