Harmonic bases for generalized coinvariant algebras

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Outline

1. The classical coinvariant algebra R_n and its harmonic space V_n

2. The generalized coinvariant algebra $R_{n,\lambda}$

3. Describe the harmonic space and construct a harmonic basis for $R_{n,\lambda}$.

Classical coinvariant algebra

Let I_n be an ideal of $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$ defined as

$$I_n := \langle e_1, \ldots, e_n \rangle$$

where e_d is the elementary symmetric polynomial of degree d.

The classical coinvariant ring R_n is the associated quotient ring

$$R_n := \mathbb{Q}[\mathbf{x}_n]/I_n$$

Some properties of R_n

1. **Artin**: The following set of monomials:

$$\{x_1^{i_1} \dots x_n^{i_n} : 0 \leq i_j \leq n - j\}$$

descends to a basis of R_n.

2. **Chevalley**: R_n is isomorphic to the regular representation $\mathbb{Q}[\mathfrak{S}_n]$ as ungraded $\mathfrak{S}_n$-modules.

3. **Lusztig-Stanley**:

$$\mathrm{grFrob}(R_n; q) = \sum_{w = w_1 \dots w_n} q^{maj(w)} x_{w_1} \dots x_{w_n}$$

Defining the harmonic space

Take $f \in \mathbb{Q}[\mathbf{x}_n]$. Let ∂f be the differential operator

$$\partial f := f(\partial/\partial x_1, \ldots \partial/\partial x_n)$$

Then $\mathbb{Q}[\mathbf{x}_n]$ acts on itself by:

$$f \odot g := (\partial f)(g)$$

We also define an inner product of $\mathbb{Q}[\mathbf{x}_n]$:

$$\langle f, g \rangle := \text{constant term of } f \odot g$$

Defining the harmonic space

Let $I \subset \mathbb{Q}[\mathbf{x}_n]$ be a homogeneous ideal. Its harmonic space V is defined as:

$$V := I^{\perp} = \{g \in \mathbb{Q}[\mathbf{x}_n] \,:\, \langle f, g \rangle = 0 \text{ for all } f \in I\}$$

A basis of V is called a *harmonic basis*.

Fact: If I is $\mathfrak{S}_n$-invariant, then $\mathbb{Q}[\mathbf{x}_n]/I \cong V$ as graded $\mathfrak{S}_n$-modules.

Now, let V_n be the harmonic space associated to R_n.

Motivating V_n

Why we want to study V_n, instead of R_n?

Answer: It is hard to determine whether $f + I_n = 0$ for a given $f \in \mathbb{Q}[\mathbf{x}_n]$. We can avoid this challenge by studying V_n. Elements of V_n are polynomials, not cosets.

Describe V_n

Fact: V_n is the smallest space that contains δ_n and is closed under $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Here, δ_n is the *Vandermonde determinant*:

$$\delta_n := \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Fact: The following is a basis of V_n.

$$\{(x_1^{c_1} \cdots x_n^{c_n}) \odot \delta_n \ : \ 0 \leq c_i \leq n - i\}.$$

From R_n to $R_{n,\lambda}$

Sean Griffin generalized R_n to $R_{n,\lambda}$. Let $k \leq n$ be nonnegative integers and let λ be a partition of k with s parts. Then let $I_{n,\lambda} \subseteq \mathbb{Q}[\mathbf{x}_n]$ be the ideal generated by $x_1^s, \ldots, x_n^s$ and $e_d(S)$, where the range of S and d will be illustrated in the next example.

Let $R_{n,\lambda} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$ be the associated quotient ring. Let $V_{n,\lambda}$ be the harmonic space.

An example of $I_{n,\lambda}$

Assume $n = 9, k = 7, s = 4$, and $\lambda = (3, 2, 2, 0)$.
$I_{9,(3,2,2,0)}$ is generated by $x_1^4, \ldots, x_9^4$ together with: $e_d(S)$, where possible d, S are:

$$\begin{array}{|c|c|c|}\hline 9 & 8 & 7 \\ \hline 6 & 5 \\ \cline{1-2} 4 & 3 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline \cdot & 8 & 7 \\ \hline \cdot & 6 \\ \cline{1-2} \cdot & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline \cdot & \cdot & 7 \\ \hline \cdot & \cdot \\ \cline{1-2} \cdot & \cdot \\ \cline{1-2}\end{array}$$

$$|S| = 9 \qquad |S| = 8 \qquad |S| = 7$$

Some special cases of $R_{n,\lambda}$

1. When $k = s = n$ and $\lambda = (1^n)$, then $R_{n,\lambda} = R_n$.
2. When $k = n$, λ is a partition of n. The ring $R_{n,\lambda}$ is the *Tanisaki quotient* studied by Tanisaki and Garsia-Procesi.
3. When $\lambda = (1^k, 0^{s-k})$, the ring $R_{n,\lambda}$ was introduced by Haglund, Rhoades and Shimozono to give a representation-theoretic model for the Haglund-Remmel-Wilson Delta Conjecture

Injective tableaux

Let λ be a partition. Let $\mathrm{Inj}(\lambda; \leq n)$ be the family of tableaux of shape λ such that:

1. Each column is strictly increasing
2. No two entries are the same
3. Each entry is at most n

$\mathrm{Inj}((4,2,1,0,0); \leq 9)$ contains

2	1	3	9
5	4		
6			

Generalizing Vandermonde

For any subset $S \subseteq [n]$, define

$$\delta_S := \prod_{\substack{i,j \in S \\ i<j}} (x_i - x_j)$$

Take $T \in \mathrm{Inj}(\lambda; \leq n)$, where λ has s parts. Let $C_1, \ldots, C_r$ be columns of T. Then

$$\delta_T := \delta_{C_1} \cdots \delta_{C_r} \times \prod x_i^{s-1}$$

where the final product is over all $i \in [n]$ which do not appear in T.

δ_T example

Let T be the following element in $\mathrm{Inj}((4,2,1,0,0);\leq 9)$:

$$\begin{array}{|c|c|c|c|}\hline 2 & 1 & 3 & 9 \\ \hline 5 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

Then $C_1 = \{2,5,6\}$, and

$$\delta_{C_1} = (x_2 - x_5)(x_2 - x_6)(x_5 - x_6)$$

Then we have

$$\begin{aligned}\delta_T &= \delta_{\{2,5,6\}} \times \delta_{\{1,4\}} \times \delta_{\{3\}} \times \delta_{\{9\}} \times x_7^4 x_8^4 \\ &= (x_2 - x_5)(x_2 - x_6)(x_5 - x_6) \times (x_1 - x_4) \times 1 \times 1 \times x_7^4 x_8^4.\end{aligned}$$

Describing $V_{n,\lambda}$

Theorem ([Rhoades-Y-Zhao])

Let $k \leq n$ and λ be a partition of k. The harmonic space $V_{n,\lambda}$ is the smallest subspace of $\mathbb{Q}[\mathbf{x}_n]$ which

- *contains δ_T for any $T \in \mathrm{Inj}(\lambda, \leq n)$, and*
- *is closed under $\partial/\partial x_1, \ldots, \partial/\partial x_n$.*

When $k = n$, this statement was proved by N.Bergeron and Garsia.

A spanning set of $V_{n,\lambda}$

Goal: construct a basis of $V_{n,\lambda}$.

Fact: The following is a spanning set of $V_{n,\lambda}$:

$$\{(x_1^{b_1} \cdots x_n^{b_n}) \odot \delta_T \ : \ T \in \mathrm{Inj}(\lambda; \leq n), \ b_i \geq 0\}$$

Strategy: Extract a basis from this spanning set. To do so, we need to study some combinatorial objects.

Ordered set partition

Given $k \le n$ and a partition λ of k with s parts, let $\mathcal{OP}_{n,\lambda}$ be the family of sequences $\sigma = (B_1, \ldots, B_s)$ of subsets of $[n]$ such that $[n] = B_1 \sqcup \cdots \sqcup B_s$ and $|B_i| \ge \lambda_i$ for all i.

For example, if $n = 16$ and $\lambda = (3, 3, 2, 2, 0, 0)$, then $\mathcal{OP}_{n,\lambda}$ contains the following:

$$
\begin{array}{cccccc}
 & 14 & & & & 16 \\
9 & 10 & 15 & & \varnothing & 11 \\
5 & 8 & 12 & 13 & & \\
3 & 7 & 2 & 4 & & \\
1 & 6 & & & &
\end{array}
$$

Coinversion code of permutations

Recall that a coinversion pair of $w \in \mathfrak{S}_n$ is (i, j), where $i < j$ and j is to the right of i in one-line notation of w.

We can encode w as $(c_1, \ldots, c_n)$, where c_i counts the number of coinversion pair (i, j) in w. This is called the coinversion code of w.

For instance, if w is 31452 in one-line notation, then its coinversion code is $(3, 0, 2, 1, 0)$.

Generalizing coinversion pair

Take $\sigma \in \mathcal{OP}_{n,\lambda}$. For $1 \le i < j \le n$, we say that the pair (i,j) is a *coinversion* of σ when one of the following three conditions holds:

- i is not floating: j is to the right of i and on the same row of i.
- i is not floating: j is to the left of i and is one row below i.
- i is floating: j is to the right of i and is on the top of the container.

	14				16
9	10	15		$\varnothing$	11
5	8	12	13		
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\begin{array}{cccccc}
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9 & 10 & 15 & & \varnothing & 11 \\
\hline
\multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{13} & & \\
\hline
\multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{\color{red}2} & \multicolumn{1}{c|}{4} & & \\
\hline
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{\color{red}6} & & & & \\
\cline{1-2}
\end{array}
$$

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\cline{1-4}
\multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & & \\
\cline{1-4}
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{6} & & & & \\
\cline{1-2}
\end{array}
$$

Generalizing coinversion code

For $1 \leq i \leq n$, assume i is in p^{th} block of σ, we define c_i as

$$\begin{cases} |\{i < j \,:\, (i,j) \text{ is a coinversion of } \sigma\}| & i \text{ not floating} \\ |\{i < j \,:\, (i,j) \text{ is a coinversion of } \sigma\}| + (p-1) & \text{otherwise} \end{cases}$$

The *coinversion code* of σ is given by $\texttt{code}(\sigma) := (c_1, \ldots, c_n)$.

$$\begin{array}{cccccc}
 & 14 & & & & 16 \\
9 & 10 & 15 & & \varnothing & 11 \\
\boxed{5} & \boxed{8} & \boxed{12} & \boxed{13} & & \\
\boxed{3} & \boxed{7} & \boxed{2} & \boxed{4} & & \\
\boxed{1} & \boxed{6} & & & &
\end{array}$$

$$\texttt{code}(\sigma) = (1, 2, 2, 1, 3, 0, 0, 2, 2, 3, 5, 1, 0, 1, 2, 5).$$

maxcode

For $1 \leq i \leq n$, we define a_i as

$$\begin{cases} |\{i < j \ : \ i, j \text{ are on the same row}\}| & i \text{ not floating} \\ s - 1 & \text{otherwise} \end{cases}$$

The *max code* of σ is given by $\texttt{maxcode}(\sigma) := (a_1, \ldots, a_n)$.

$$\begin{array}{cccccc} & 14 & & & & 16 \\ 9 & 10 & 15 & & \varnothing & 11 \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 5 & 8 & 12 & 13 \\ \hline 3 & 7 & 2 & 4 \\ \hline 1 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

$$\texttt{maxcode}(\sigma) = (1, 3, 2, 1, 3, 0, 0, 2, 5, 5, 5, 1, 0, 5, 5, 5)$$

$T(\sigma)$ and δ_σ

Let $T(\sigma)$ be the element in $\mathrm{Inj}(\lambda; \leq n)$ whose column i consists of elements on row i of σ.

$$\sigma = \begin{array}{cccccc} & 14 & & & & 16 \\ 9 & 10 & 15 & & \varnothing & 11 \\ \hline \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{13} & & \\ \cline{1-4} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & & \\ \cline{1-4} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{6} & & & & \\ \cline{1-2} \end{array} \qquad T(\sigma) = \begin{array}{|c|c|c|} \hline 5 & 2 & 1 \\ \hline 8 & 3 & 6 \\ \hline 12 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 13 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

Define δ_σ by the rule

$$\delta_\sigma := (x_1^{a_1 - c_1} \cdots x_n^{a_n - c_n}) \odot \delta_{T(\sigma)}$$

where $\mathrm{code}(\sigma) = (c_1, \ldots, c_n)$ and $\mathtt{maxcode}(\sigma) = (a_1, \ldots, a_n)$

δ_σ example

$$\sigma = \begin{array}{cccccc} & 14 & & & & 16 \\ 9 & 10 & 15 & & \varnothing & 11 \\ \boxed{5} & \boxed{8} & \boxed{12} & \boxed{13} & & \\ \boxed{3} & \boxed{7} & \boxed{2} & \boxed{4} & & \\ \boxed{1} & \boxed{6} & & & & \end{array} \qquad T(\sigma) = \begin{array}{ccc} \boxed{5} & \boxed{2} & \boxed{1} \\ \boxed{8} & \boxed{3} & \boxed{6} \\ \boxed{12} & \boxed{4} & \\ \boxed{13} & \boxed{7} & \end{array}$$

$$\mathtt{maxcode}(\sigma) = (1,3,2,1,3,0,0,2,5,5,5,1,0,5,5,5)$$

$$\mathtt{code}(\sigma) = (1,2,2,1,3,0,0,2,2,3,5,1,0,1,2,5)$$

Finally, we have:

$$\delta_\sigma = (x_1^0 x_2^1 x_3^0 x_4^0 x_5^0 x_6^0 x_7^0 x_8^0 x_9^3 x_{10}^2 x_{11}^0 x_{12}^0 x_{13}^0 x_{14}^4 x_{15}^3 x_{16}^0) \odot \delta_{T(\sigma)}.$$

Harmonic Basis

Theorem ([Rhoades-Y-Zhao])

Let $k \leq n$ be positive integers and let λ be a partition of k with s parts. The set

$$\{\delta_\sigma \,:\, \sigma \in \mathcal{OP}_{n,\lambda}\}$$

is a harmonic basis of $R_{n,\lambda}$. The lexicographical leading term of δ_σ has exponent sequence $\texttt{code}(\sigma)$.

This result implies a combinatorial formula for the Hilbert series of $R_{n,\lambda}$:

$$\mathrm{Hilb}(R_{n,\lambda}; q) = \sum_{\sigma \in \mathcal{OP}_{n,\lambda}} q^{sum(\texttt{code}(\sigma))}.$$

A future direction

We can introduce a new set of variables $y_1, \ldots, y_n$ to $V_{n,\lambda}$. Define $DV_{n,\lambda}$ to be the smallest space such that:

1. It contains contains δ_T for any $T \in \mathrm{Inj}(\lambda, \leq n)$
2. It is closed under $\partial/\partial x_1, \ldots, \partial/\partial x_n$ and $\partial/\partial y_1, \ldots, \partial/\partial y_n$
3. It is closed under $y_1(\partial/\partial x_1) + \cdots + y_n(\partial/\partial x_n)$

Question: What is its Bigraded Frobenius image?

Haiman solved the special case: $\lambda = (1^n)$.

Thanks for listening!!

- B. Rhoades, T. Yu, and Z. Zhao. Harmonic bases for generalized coinvariant algebras. *Electronic Journal of Combinatorics*, **4 (4)** (2020))