Harmonic bases for generalized coinvariant algebras

Tianyi Yu
(Joint with Brendon Rhoades and Zehong Zhao)

UCSD
Outline

1. The classical coinvariant algebra $R_n$ and its harmonic space $V_n$

2. The generalized coinvariant algebra $R_{n,\lambda}$

3. Describe the harmonic space and construct a harmonic basis for $R_{n,\lambda}$. 
Classical coinvariant algebra

Let $I_n$ be an ideal of $\mathbb{Q}[x_n] := \mathbb{Q}[x_1, \ldots, x_n]$ defined as

$$I_n := \langle e_1, \ldots, e_n \rangle$$

where $e_d$ is the elementary symmetric polynomial of degree $d$.

The classical coinvariant ring $R_n$ is the associated quotient ring

$$R_n := \mathbb{Q}[x_n]/I_n$$
Some properties of $R_n$

1. **Artin**: The following set of monomials:

   \[
   \{ x_1^{i_1} \ldots x_n^{i_n} : 0 \leq i_j \leq n - j \}
   \]

   descends to a basis of $R_n$.

2. **Chevalley**: $R_n$ is isomorphic to the regular representation $\mathbb{Q}[\mathfrak{S}_n]$ as ungraded $\mathfrak{S}_n$-modules.

3. **Lusztig-Stanley**:

   \[
   \text{grFrob}(R_n; q) = \sum_{w = w_1 \ldots w_n} q^{\text{maj}(w)} x_{w_1} \ldots x_{w_n}
   \]
Defining the harmonic space

Take $f \in \mathbb{Q}[x_n]$. Let $\partial f$ be the differential operator

$$\partial f := f(\partial/\partial x_1, \ldots \partial/\partial x_n)$$

Then $\mathbb{Q}[x_n]$ acts on itself by:

$$f \odot g := (\partial f)(g)$$

We also define an inner product of $\mathbb{Q}[x_n]$: 

$$\langle f, g \rangle := \text{constant term of } f \odot g$$
Defining the harmonic space

Let \( I \subset \mathbb{Q}[x_n] \) be a homogeneous ideal. Its harmonic space \( V \) is defined as:

\[
V := I^\perp = \{ g \in \mathbb{Q}[x_n] : \langle f, g \rangle = 0 \text{ for all } f \in I \}
\]

A basis of \( V \) is called a *harmonic basis*.

Fact: If \( I \) is \( S_n \)-invariant, then \( \mathbb{Q}[x_n]/I \cong V \) as graded \( S_n \)-modules.

Now, let \( V_n \) be the harmonic space associated to \( R_n \).
Motivating $V_n$

Why we want to study $V_n$, instead of $R_n$?

Answer: It is hard to determine whether $f + I_n = 0$ for a given $f \in \mathbb{Q}[x_n]$. We can avoid this challenge by studying $V_n$. Elements of $V_n$ are polynomials, not cosets.
Describe $V_n$

**Fact:** $V_n$ is the smallest space that contains $\delta_n$ and is closed under $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Here, $\delta_n$ is the *Vandermonde determinant*:

$$\delta_n := \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

**Fact:** The following is a basis of $V_n$.

$$\{(x_1^{c_1} \cdots x_n^{c_n}) \odot \delta_n : 0 \leq c_i \leq n - i\}.$$
From $R_n$ to $R_{n,\lambda}$

Sean Griffin generalized $R_n$ to $R_{n,\lambda}$. Let $k \leq n$ be nonnegative integers and let $\lambda$ be a partition of $k$ with $s$ parts. Then let $I_{n,\lambda} \subseteq \mathbb{Q}[x_n]$ be the ideal generated by $x_1^s, \ldots, x_n^s$ and $e_d(S)$, where the range of $S$ and $d$ will be illustrated in the next example.

Let $R_{n,\lambda} := \mathbb{Q}[x_n]/I_{n,\lambda}$ be the associated quotient ring. Let $V_{n,\lambda}$ be the harmonic space.
An example of $I_{n,\lambda}$

Assume $n = 9$, $k = 7$, $s = 4$, and $\lambda = (3, 2, 2, 0)$.

$I_{9,(3,2,2,0)}$ is generated by $x_1^4, \ldots, x_9^4$ together with: $e_d(S)$, where possible $d, S$ are:

\[
\begin{array}{ccc}
9 & 8 & 7 \\
6 & 5 \\
4 & 3 \\
\end{array}
\quad
\begin{array}{ccc}
\cdot & 8 & 7 \\
\cdot & 6 \\
\cdot & 5 \\
\end{array}
\quad
\begin{array}{ccc}
\cdot & \cdot & 7 \\
\cdot & \cdot \\
\cdot & \cdot \\
\end{array}
\]

$|S| = 9 \quad |S| = 8 \quad |S| = 7$
Some special cases of $R_{n,\lambda}$

1. When $k = s = n$ and $\lambda = (1^n)$, then $R_{n,\lambda} = R_n$.

2. When $k = n$, $\lambda$ is a partition of $n$. The ring $R_{n,\lambda}$ is the Tanisaki quotient studied by Tanisaki and Garsia-Procesi.

3. When $\lambda = (1^k, 0^{s-k})$, the ring $R_{n,\lambda}$ was introduced by Haglund, Rhoades and Shimozono to give a representation-theoretic model for the Haglund-Remmel-Wilson Delta Conjecture.
Injective tableaux

Let $\lambda$ be a partition. Let $\text{Inj}(\lambda; \leq n)$ be the family of tableaux of shape $\lambda$ such that:

1. Each column is strictly increasing
2. No two entries are the same
3. Each entry is at most $n$

$\text{Inj}((4, 2, 1, 0, 0); \leq 9)$ contains

```
  2 1 3 9
  5 4
  6
```
For any subset $S \subseteq [n]$, define

$$\delta_S := \prod_{i,j \in S, i < j} (x_i - x_j)$$

Take $T \in \text{Inj}(\lambda; \leq n)$, where $\lambda$ has $s$ parts. Let $C_1, \ldots, C_r$ be columns of $T$. Then

$$\delta_T := \delta_{C_1} \cdots \delta_{C_r} \times \prod x_i^{s-1}$$

where the final product is over all $i \in [n]$ which do not appear in $T$. 

**Generalizing Vandermonde**
Let $T$ be the following element in \( \text{Inj}((4, 2, 1, 0, 0); \leq 9) $:

\[
\begin{array}{cccc}
2 & 1 & 3 & 9 \\
5 & 4 \\
6
\end{array}
\]

Then $C_1 = \{2, 5, 6\}$, and

\[
\delta_{C_1} = (x_2 - x_5)(x_2 - x_6)(x_5 - x_6)
\]

Then we have

\[
\delta_T = \delta_{\{2,5,6\}} \times \delta_{\{1,4\}} \times \delta_{\{3\}} \times \delta_{\{9\}} \times x_7^4 x_8^4
\]

\[
= (x_2 - x_5)(x_2 - x_6)(x_5 - x_6) \times (x_1 - x_4) \times 1 \times 1 \times x_7^4 x_8^4.
\]
Describing $V_{n,\lambda}$

**Theorem ([Rhoades-Y-Zhao])**

Let $k \leq n$ and $\lambda$ be a partition of $k$. The harmonic space $V_{n,\lambda}$ is the smallest subspace of $\mathbb{Q}[x_n]$ which

- contains $\delta_T$ for any $T \in \text{Inj}(\lambda, \leq n)$, and
- is closed under $\partial/\partial x_1, \ldots, \partial/\partial x_n$.

When $k = n$, this statement was proved by N.Bergeron and Garsia.
A spanning set of $V_{n,\lambda}$

**Goal:** construct a basis of $V_{n,\lambda}$.

**Fact:** The following is a spanning set of $V_{n,\lambda}$:

$$\{(x_1^{b_1} \cdots x_n^{b_n}) \otimes \delta_T : T \in \text{Inj}(\lambda; \leq n), \ b_i \geq 0\}$$

**Strategy:** Extract a basis from this spanning set. To do so, we need to study some combinatorial objects.
Ordered set partition

Given $k \leq n$ and a partition $\lambda$ of $k$ with $s$ parts, let $\mathcal{OP}_{n,\lambda}$ be the family of sequences $\sigma = (B_1, \ldots, B_s)$ of subsets of $[n]$ such that $[n] = B_1 \sqcup \cdots \sqcup B_s$ and $|B_i| \geq \lambda_i$ for all $i$.

For example, if $n = 16$ and $\lambda = (3, 3, 2, 2, 0, 0)$, then $\mathcal{OP}_{n,\lambda}$ contains the following:

$$
\begin{array}{cccc}
14 & 16 \\
9 & 10 & 15 & \emptyset & 11 \\
5 & 8 & 12 & 13 \\
3 & 7 & 2 & 4 \\
1 & 6
\end{array}
$$
Coinversion code of permutations

Recall that a coinversion pair of \( w \in S_n \) is \((i, j)\), where \( i < j \) and \( j \) is to the right of \( i \) in one-line notation of \( w \).

We can encode \( w \) as \((c_1, \ldots, c_n)\), where \( c_i \) counts the number of coinversion pair \((i, j)\) in \( w \). This is called the coinversion code of \( w \).

For instance, if \( w \) is 31452 in one-line notation, then its coinversion code is \((3, 0, 2, 1, 0)\).
Generalizing coinversion pair

Take $\sigma \in \mathcal{OP}_{n,\lambda}$. For $1 \leq i < j \leq n$, we say that the pair $(i,j)$ is a coinversion of $\sigma$ when one of the following three conditions holds:

- $i$ is not floating: $j$ is to the right of $i$ and on the same row of $i$.
- $i$ is not floating: $j$ is to the left of $i$ and is one row below $i$.
- $i$ is floating: $j$ is to the right of $i$ and is on the top of the container.
Generalizing coinversion pair

Take $\sigma \in \mathcal{OP}_{n,\lambda}$. For $1 \leq i < j \leq n$, we say that the pair $(i, j)$ is a coinversion of $\sigma$ when one of the following three conditions holds:

- $i$ is not floating: $j$ is to the right of $i$ and on the same row of $i$.
- $i$ is not floating: $j$ is to the left of $i$ and is one row below $i$.
- $i$ is floating: $j$ is to the right of $i$ and is on the top of the container.

\[
\begin{array}{cccc}
14 & 15 & \emptyset & 11 \\
9 & 10 & 12 & 13 \\
5 & 8 & 2 & 4 \\
1 & 6
\end{array}
\]
Generalizing coinversion pair

Take \( \sigma \in \mathcal{OP}_{n, \lambda} \). For \( 1 \leq i < j \leq n \), we say that the pair \((i, j)\) is a coinversion of \( \sigma \) when one of the following three conditions holds:

- \( i \) is not floating: \( j \) is to the right of \( i \) and on the same row of \( i \).
- \( i \) is not floating: \( j \) is to the left of \( i \) and is one row below \( i \).
- \( i \) is floating: \( j \) is to the right of \( i \) and is on the top of the container.

\[
\begin{array}{cccc}
14 & 16 & 15 & 10 \\
9 & 10 & 5 & 8 \\
3 & 7 & 2 & 4 \\
1 & 6 & \emptyset & 11 \\
\end{array}
\]
Generalizing coinversion code

For $1 \leq i \leq n$, assume $i$ is in $p^{th}$ block of $\sigma$, we define $c_i$ as

$$
\begin{cases}
|\{i < j : (i, j) \text{ is a coinversion of } \sigma\}| & \text{if } i \text{ not floating} \\
|\{i < j : (i, j) \text{ is a coinversion of } \sigma\}| + (p - 1) & \text{otherwise}
\end{cases}
$$

The coinversion code of $\sigma$ is given by $\text{code}(\sigma) := (c_1, \ldots, c_n)$.

\[
\begin{array}{cccccc}
14 & 16 \\
9 & 10 & 15 & \emptyset & 11 \\
5 & 8 & 12 & 13 \\
3 & 7 & 2 & 4 \\
1 & 6
\end{array}
\]

\[
\text{code}(\sigma) = (1, 2, 2, 1, 3, 0, 0, 2, 2, 3, 5, 1, 0, 1, 2, 5).
\]
For $1 \leq i \leq n$, we define $a_i$ as

$$
\begin{cases}
|i < j : i, j \text{ are on the same row}| & \text{if } i \text{ not floating} \\
s - 1 & \text{otherwise}
\end{cases}
$$

The max code of $\sigma$ is given by $\text{maxcode}(\sigma) := (a_1, \ldots, a_n)$.

$$
\begin{array}{cccc}
14 & 16 \\
9 & 10 & 15 & \varnothing & 11 \\
5 & 8 & 12 & 13 \\
3 & 7 & 2 & 4 \\
1 & 6
\end{array}
$$

$\text{maxcode}(\sigma) = (1, 3, 2, 1, 3, 0, 0, 2, 5, 5, 5, 1, 0, 5, 5, 5)$
$T(\sigma)$ and $\delta_\sigma$

Let $T(\sigma)$ be the element in $\text{Inj}(\lambda; \leq n)$ whose column $i$ consists of elements on row $i$ of $\sigma$.

$$\sigma = \begin{array}{cccc}
1 & 4 & 16 & 9 \\
10 & 15 & \emptyset & 11 \\
5 & 8 & 12 & 13 \\
3 & 7 & 2 & 4 \\
1 & 6
\end{array} \quad T(\sigma) = \begin{array}{ccc}
5 & 2 & 1 \\
8 & 3 & 6 \\
12 & 4 \\
13 & 7
\end{array}$$

Define $\delta_\sigma$ by the rule

$$\delta_\sigma := (x_1^{a_1-c_1} \cdots x_n^{a_n-c_n}) \odot \delta_{T(\sigma)}$$

where $\text{code}(\sigma) = (c_1, \ldots, c_n)$ and $\text{maxcode}(\sigma) = (a_1, \ldots, a_n)$
$\delta_\sigma$ example

\[
\begin{array}{ccc}
\sigma &=& \begin{array}{cccc}
14 & 16 \\
9 & 10 & 15 & \emptyset & 11 \\
5 & 8 & 12 & 13 \\
3 & 7 & 2 & 4 \\
1 & 6 \\
\end{array} \\
T(\sigma) &=& \begin{array}{ccc}
5 & 2 & 1 \\
8 & 3 & 6 \\
12 & 4 \\
13 & 7 \\
\end{array}
\end{array}
\]

\[\text{maxcode}(\sigma) = (1, 3, 2, 1, 3, 0, 0, 2, 5, 5, 5, 1, 0, 5, 5, 5)\]

\[\text{code}(\sigma) = (1, 2, 2, 1, 3, 0, 0, 2, 2, 3, 5, 1, 0, 1, 2, 5)\]

Finally, we have:

\[
\delta_\sigma = (x_1^0 x_2^1 x_3^0 x_4^0 x_5^0 x_6^0 x_7^0 x_8^0 x_9^3 x_{10}^2 x_{11}^0 x_{12}^0 x_{13}^0 x_{14}^4 x_{15}^3 x_{16}^0) \odot \delta T(\sigma).
\]
Harmonic Basis

**Theorem ([Rhoades-Y-Zhao])**

Let $k \leq n$ be positive integers and let $\lambda$ be a partition of $k$ with $s$ parts. The set

$$\{\delta_\sigma : \sigma \in \mathcal{OP}_{n,\lambda}\}$$

is a harmonic basis of $R_{n,\lambda}$. The lexicographical leading term of $\delta_\sigma$ has exponent sequence $\text{code}(\sigma)$.

This result implies a combinatorial formula for the Hilbert series of $R_{n,\lambda}$:

$$\text{Hilb}(R_{n,\lambda}; q) = \sum_{\sigma \in \mathcal{OP}_{n,\lambda}} q^{\text{sum}(\text{code}(\sigma))}.$$
A future direction

We can introduce a new set of variables \( y_1, \ldots, y_n \) to \( V_{n, \lambda} \). Define \( DV_{n, \lambda} \) to be the smallest space such that:

1. It contains \( \delta_T \) for any \( T \in \text{Inj}(\lambda, \leq n) \)

2. It is closed under \( \partial/\partial x_1, \ldots, \partial/\partial x_n \) and \( \partial/\partial y_1, \ldots, \partial/\partial y_n \)

3. It is closed under \( y_1(\partial/\partial x_1) + \cdots + y_n(\partial/\partial x_n) \)

**Question:** What is its Bigraded Frobenius image?

Haiman solved the special case: \( \lambda = (1^n) \).
Thanks for listening!!