Grothendieck-to-Lascoux expansions

Mark Shimozono (Virginia Tech) and Tianyi Yu (UCSD)

September, 2021

Outline

1. Introducing 8 polynomials

2. Several combinatorial formulas

3. The Grothendieck-to-Lascoux expansions

Operators

Define operators on $\mathbb{Z}[\beta][x_1, x_2, \dots, x_n]$:

$$\begin{aligned}
\partial_i(f) &= (x_i - x_{i+1})^{-1}(f - s_i f) \\
\pi_i(f) &= \partial_i(x_i f) \\
\partial_i^{(\beta)}(f) &= \partial_i(f + \beta x_{i+1} f) \\
\pi_i^{(\beta)}(f) &= \partial_i^{(\beta)}(x_i f).
\end{aligned}$$

They satisfy the braid relations.

Lascoux polynomials and key polynomials

For weak composition α,

$$\mathfrak{L}_{\alpha}^{(\beta)} := \begin{cases} x^{\alpha} & \text{if } \alpha \text{ is a partition} \\ \pi_i^{(\beta)} \mathfrak{L}_{s_i\alpha}^{(\beta)} & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

$$\kappa_{\alpha} := \begin{cases} x^{\alpha} & \text{if } \alpha \text{ is a partition} \\ \pi_i(\kappa_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

Fact: $\kappa_{\alpha} = \mathfrak{L}_{\alpha}^{(\beta)}|_{\beta=0}$.

- $\mathfrak{L}_{210}^{(\beta)} = x_1^2 x_2$
- $\mathfrak{L}_{120}^{(\beta)} = \pi_1^{(\beta)}(\mathfrak{L}_{210}^{(\beta)}) = x_1^2 x_2 + x_1 x_2^2 + \beta x_1^2 x_2^2$

β-Grothendieck Polynomials and Schubert Polynomials

For $w \in S_{n+1}$,

$$\mathfrak{G}_w^{(\beta)} := \begin{cases} x_1^n x_2^{n-1} \cdots x_n & \text{if } w = (n+1, n, \ldots, 1) \\ \partial_i^{(\beta)}(\mathfrak{G}_{ws_i}^{(\beta)}) & \text{if } ws_i > w. \end{cases}$$

$$\mathfrak{S}_w := \begin{cases} x_1^n x_2^{n-1} \cdots x_n & \text{if } w = (n+1, n, \ldots, 1) \\ \partial_i(\mathfrak{S}_{ws_i}) & \text{if } ws_i > w. \end{cases}$$

Fact: $\mathfrak{S}_w = \mathfrak{G}_w^{(\beta)}|_{\beta=0}$.

- $\mathfrak{G}_{321}^{(\beta)} = x_1^2 x_2$
- $\mathfrak{G}_{312}^{(\beta)} = \partial_2^{(\beta)}(\mathfrak{G}_{321}^{(\beta)}) = x_1^2$
- $\mathfrak{G}_{132}^{(\beta)} = \partial_1^{(\beta)}(\mathfrak{G}_{312}^{(\beta)}) = x_1 + x_2 + \beta x_1 x_2$

Symmetrization

For $w \in S_n$, let $\pi_w^{(\beta)} = \pi_{i_1}^{(\beta)} \cdots \pi_{i_k}^{(\beta)}$, where $s_{i_1} \cdots s_{i_k} = w$.
Let $w_0 := (n, n-1, \ldots, 1)$.

$$\pi_{w_0}^{(\beta)}(\mathfrak{L}_\alpha^{(\beta)}) = G_{\alpha^+}^{(\beta)} \qquad \text{(Grass. Symm. Groth. polynomials)}$$

$$\pi_{w_0}(\kappa_\alpha) = s_{\alpha^+} \qquad \text{(Schur polynomials)}$$

$$\pi_{w_0}^{(\beta)}(\mathfrak{G}_w^{(\beta)}) = G_w^{(\beta)} \qquad \text{(Symm. Groth. polynomials)}$$

$$\pi_{w_0}(\mathfrak{S}_w) = F_w \qquad \text{(Stanley Symmetric Functions)}$$

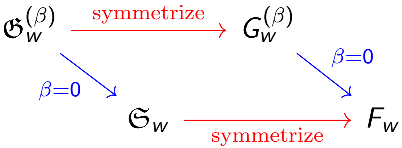
Fact: When α is weakly increasing,

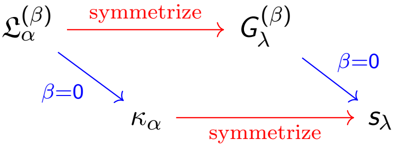
$$\mathfrak{L}_\alpha^{(\beta)} = G_{\alpha^+}^{(\beta)} \qquad \text{and} \qquad \kappa_\alpha = s_{\alpha^+}$$

Quick review

polynomial	has β	symmetric	index
$\mathfrak{L}_{\alpha}^{(\beta)}$	✓	×	α
κ_{α}	×	×	α
$\mathfrak{G}_{w}^{(\beta)}$	✓	×	w
$\mathfrak{S}_{w}$	×	×	w
$G_{\lambda}^{(\beta)}$	✓	✓	λ
s_{λ}	×	✓	λ
$G_{w}^{(\beta)}$	✓	✓	w
F_{w}	×	✓	w

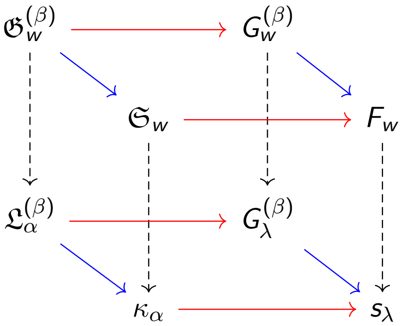
Quick review





Next, we connect these two diagrams.

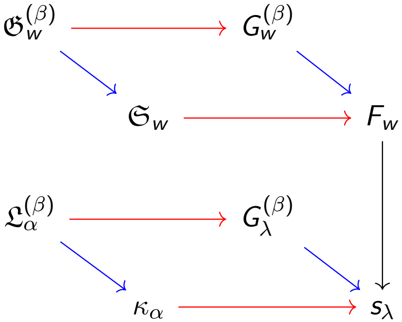
Expansions



$\mathfrak{G}_w^{(\beta)}$
$G_w^{(\beta)}$
$\mathfrak{S}_w$
F_w
$\mathfrak{L}_\alpha^{(\beta)}$
$G_\lambda^{(\beta)}$
κ_α
s_λ

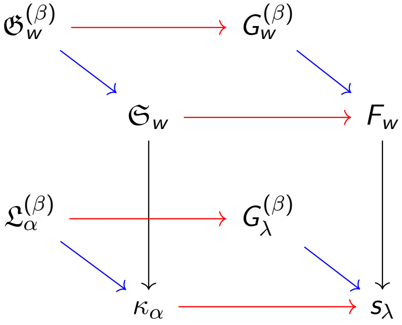
F_w into s_λ

In 1987, Edelman and Greene expanded F_w into s_λ.



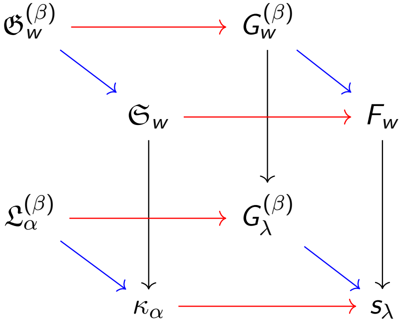
$\mathfrak{S}_w$ into κ_α

In 1995, Reiner and Shimozono expanded $\mathfrak{S}_w$ into κ_α. This expansion was first stated by Lascoux and Schützenberger in 1989.



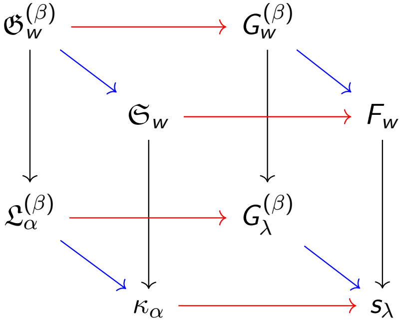
$G_w^{(\beta)}$ into $G_\lambda^{(\beta)}$

In 2008, Buch, Kresch, Shimozono, Tamvakis, Yong expanded $G_w^{(\beta)}$ into $G_\lambda^{(\beta)}$.



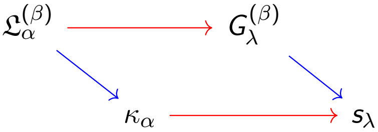
$\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$

In 2021, Shimozono and Yu expanded $\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$. This expansion was conjectured by Reiner and Yong.



Some tableaux formulas for $\mathfrak{L}_\alpha^{(\beta)}$

- When $\beta = 0$, reverse semistandard Young tableaux rule with key condition [Lascoux, Schützenberger 1980].
- When α is weakly increasing, reversed set-valued tableaux rule [Buch 2002].
- Reverse set-valued tableaux with key condition. Implicit in [Buciumas, Scrimshaw, Weber 2020]; rediscovered by [Shimozono,Y 2021]



Other combinatorial formulas for $\mathfrak{L}_\alpha^{(\beta)}$

- K-Kohnert diagrams. Conjectured [Ross, Yong 2015]. Rectangle case proved [Pechenik, Scrimshaw 2019]
- Set-valued skyline fillings. Conjectured [Monical, 2017] proved [Buciumas, Scrimshaw, Weber 2020]
- Set-valued tableaux with K-crystal Lusztig involution and key condition. Conjectured [Pechenik, Scrimshaw 2020] proved [BSW]
- Set-valued tableaux with key condition. [Y (In preparation)]

Keys

A *key* is a reversed semistandard Young tableau (RSSYT) where each number in column j is also in column $j-1$.
Keys are in bijection with weak compositions:

$$(1,0,3,2) \Longleftrightarrow \begin{array}{|c|c|c|}\hline 4 & 4 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

Let $\mathrm{key}(\cdot)$ send a weak composition to its corresponding key. Its inverse is $\mathrm{wt}(\cdot)$.

Antirectification

The Knuth equivalence $\equiv$ is an equivalence relation on words.

Fact: For each RSSYT T, exists unique $T^{\searrow}$ of anti-normal shape such that

$$\mathrm{rev}(\mathrm{word}(T)) \equiv \mathrm{rev}(\mathrm{word}(T^{\searrow})).$$

They can be found by jeu-de-taquin (jdt).

Left keys

Let T be a normal RSSYT. Column j of $K_-(T)$ is the first column of $T^{\searrow}_{\leq j}$.

$$\begin{array}{|c|c|c|}\hline 5&4&1\\\hline 3&3\\\cline{1-2} 2\\\cline{1-1}\end{array} \xrightarrow{K_-} \begin{array}{|c|c|c|}\hline 5&5&3\\\hline 3&3\\\cline{1-2} 2\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|}\hline 5\\\hline 3\\\hline 2\\\hline\end{array} \to \begin{array}{|c|}\hline \mathbf{5}\\\hline \mathbf{3}\\\hline \mathbf{2}\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 5&4\\\hline 3&3\\\hline 2\\\cline{1-1}\end{array} \to \begin{array}{cc}\cline{2-2} &\multicolumn{1}{|c|}{4}\\\hline \multicolumn{1}{|c|}{\mathbf{5}}&\multicolumn{1}{c|}{3}\\\hline \multicolumn{1}{|c|}{\mathbf{3}}&\multicolumn{1}{c|}{2}\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 5&4&1\\\hline 3&3\\\cline{1-2} 2\\\cline{1-1}\end{array} \to \begin{array}{ccc}\cline{3-3} &&\multicolumn{1}{|c|}{4}\\\cline{2-3} &\multicolumn{1}{|c|}{5}&\multicolumn{1}{c|}{3}\\\hline \multicolumn{1}{|c|}{\mathbf{3}}&\multicolumn{1}{c|}{2}&\multicolumn{1}{c|}{1}\\\hline\end{array}$$

Right keys

Let T be a normal RSSYT. Column j of $K_+(T)$ is the last column of $T^{\searrow}_{\geq j}$.

5	4	1
3	3	
2		

$\xrightarrow{K_+}$

4	4	1
3	1	
1		

5	4	1
3	3	
2		

$\rightarrow$

		4
	5	**3**
3	2	**1**

4	1
3	

$\rightarrow$

	4
3	**1**

1

$\rightarrow$

1

RSSYT rule for κ_α

Theorem (Lascoux, Schützenberger, 1980)

$$\kappa_\alpha = \sum_T x^{\mathrm{wt}(T)}$$

where T is a RSSYT whose shape is α^+ and $K_-(T) \leq \mathrm{key}(\alpha)$.

Example $\kappa_{(1,0,2)}$

3	3
1	

3	2
1	

3	1
1	

2	2
1	

2	1
1	

$$\kappa_{(1,0,2)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2$$

Reversed set-valued tableaux (RSVT)

The following T is an example of a RSVT

54	4	1
3	321	
21		

Then $\mathrm{wt}(T) = (3, 2, 2, 2, 1)$, $\mathrm{ex}(T) = 4$, and max(T) is

5	4	1
3	3	
2		

RSVT rule for $G_\lambda^{(\beta)}$

Theorem (Buch 2002)

Let λ be a partition with at most n parts.

$$G_\lambda^{(\beta)} = \sum_T \beta^{\mathrm{ex}(T)} x^{\mathrm{wt}(T)}$$

where T is a RSVT with shape λ s.t. its entries are subsets of $[n]$.

RSVT rule for $\mathfrak{L}_\alpha^{(\beta)}$

Theorem (Buciumas, Scrimshaw, Weber 2020; Shimozono, Y 2021)

$$\mathfrak{L}_\alpha^{(\beta)} = \sum_T \beta^{\mathrm{ex}(T)} x^{\mathrm{wt}(T)}$$

where T is a RSVT with shape α^+ s.t. $K_-(max(T)) \leq \mathrm{key}(\alpha)$.

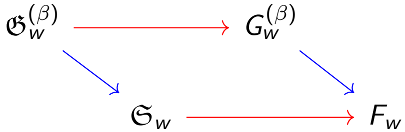
Example $\mathfrak{L}^{(\beta)}_{(1,0,2)}$

$\begin{array}{\|c\|c\|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$
$\begin{array}{\|c\|c\|}\hline 3 & 31 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 3 & 21 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 32 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 21 \\ \hline 1 \\ \cline{1-1}\end{array}$	
$\begin{array}{\|c\|c\|}\hline 3 & 32 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 32 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$			
$\begin{array}{\|c\|c\|}\hline 3 & 321 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 32 & 21 \\ \hline 1 \\ \cline{1-1}\end{array}$			

$$\begin{aligned}\mathfrak{L}^{(\beta)}_{(1,0,2)} &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 \\ &+ \beta(x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_3^2 + x_1x_2x_3^2) \\ &+ \beta^2(x_1^2x_2^2x_3 + x_1^2x_2x_3^2)\end{aligned}$$

Compatible word rules

- Pipedream rules for $\mathfrak{S}_w$. Pipedreams are in bijection with compatible words. [Billey, Jockusch, Stanley 1993] [N. Bergeron, Billey 1993]
- Compatible word rule for $G_w^{(\beta)}$. [Fomin, Kirillov 1994]
- Compatible word rule for $\mathfrak{G}_w^{(\beta)}$ with a "bounded" condition. [Fomin, Kirillov 1994]



Compatible words

Definition (Billey, Jockusch, Stanley 1993)

A pair of words (a, i) *with the same length is compatible if they satisfy*

- i *is weakly decreasing*
- $i_j = i_{j+1}$ *implies* $a_j < a_{j+1}$.

A compatible pair (a, i) is *bounded* if $a_j \geq i_j$ for all j.

0-Hecke equivalence

$\mathbb{Z}_{>0}^*$: Free monoid of words in alphabet $\mathbb{Z}_{>0}$
The 0-Hecke equivalence $\equiv_H$ on $\mathbb{Z}_{>0}^*$ is generated by:

$$
\begin{aligned}
a(a+1)a &\equiv_H (a+1)a(a+1) \\
aa &\equiv_H a \\
ab &\equiv_H ba \qquad \text{for } |a-b| \geq 2.
\end{aligned}
$$

$\mathbb{Z}_{>0}^*$ acts on S_+:

$$
i \circ w = \begin{cases} s_i w & \text{if } \ell(s_i w) > \ell(w). \\ w & \text{otherwise.} \end{cases}
$$

Let $[b]_H := b \circ \mathrm{id} \in S_+$.

Combinatorial formula for $\mathfrak{G}_w^{(\beta)}$ and $G_w^{(\beta)}$

Theorem (Fomin, Kirillov 1994)

$$\mathfrak{G}_w^{(\beta)}(x_1, \ldots, x_n) = \sum_{\substack{(a,i) \text{ compatible} \\ (a,i) \text{ bounded} \\ [a]_H = w^{-1}}} x^{\mathrm{wt}(i)} \beta^{\ell(i) - \ell(w)},$$

$$G_w^{(\beta)}(x_1, \ldots, x_n) = \sum_{\substack{(a,i) \text{ compatible} \\ [a]_H = w^{-1} \\ i_j \le n}} x^{\mathrm{wt}(i)} \beta^{\ell(i) - \ell(w)}.$$

Quick review

$$
\begin{array}{ccc}
\mathfrak{G}_w^{(\beta)} & \xrightarrow{\text{symmetrize}} & G_w^{(\beta)} \\
\text{expand} \downarrow & & \downarrow \text{expand} \\
\mathfrak{L}_\alpha^{(\beta)} & \xrightarrow[\text{symmetrize}]{} & G_\lambda^{(\beta)}
\end{array}
$$

We have

- compatible word formulas for the top two.
- RSVT formulas for the bottom two.

Q: How to connect compatible pairs and RSVT?
A: Hecke insertion!

Hecke Insertion

Let $\mathcal{C}$ be the set of all compatible pairs.
Let $\mathcal{T}$ be the set of all (P, Q) such that, P is decreasing, Q is a RSVT, and P, Q have the same shape.

Theorem (Buch,Kresch,Shimozono,Tamvakis,Yong 2008)

Hecke insertion is a bijection from $\mathcal{C}$ to $\mathcal{T}$. If we insert (a, i) and get (P, Q), then

- $[a]_H = [\mathrm{word}(P)]_H$.
- $\mathrm{wt}(i) = \mathrm{wt}(Q)$.

Expand $G_w^{(\beta)}$ into $G_\lambda^{(\beta)}$

If we Hecke insert

$$\mathcal{C}_w := \{(a,i) \in \mathcal{C} : [a]_H = w, i \text{ only has numbers in } [n]\},$$

we get $\mathcal{T}_w$, which consists of $(P,Q) \in \mathcal{T}$ such that $[\mathrm{word}(P)]_H = w$ and Q has numbers in $[n]$.

$$\begin{aligned} G_w^{(\beta)} &= \sum_{(a,i)\in\mathcal{C}_{w^{-1}}} x^{\mathrm{wt}(i)} \beta^{\ell(i)-\ell(w)} \\ &= \sum_{(P,Q)\in\mathcal{T}_{w^{-1}}} x^{\mathrm{wt}(Q)} \beta^{\mathrm{ex}(Q)+|\mathrm{shape}(Q)|-\ell(w)} \\ &= \sum_P \beta^{|\mathrm{shape}(P)|-\ell(w)} \sum_Q x^{\mathrm{wt}(Q)} \beta^{\mathrm{ex}(Q)} \\ &= \sum_P \beta^{|\mathrm{shape}(P)|-\ell(w)} G^{(\beta)}_{\mathrm{shape}(P)} \end{aligned}$$

Expand $\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$

Define

$$\mathcal{C}_w^B := \{(a, i) \in \mathcal{C} : [a]_H = w, \text{bounded}\}.$$

Recall

$$\mathfrak{G}_w^{(\beta)} = \sum_{(a,i)\in\mathcal{C}_{w^{-1}}^B} x^{\mathrm{wt}(i)} \beta^{\ell(i)-\ell(w)}$$

If we Hecke insert $\mathcal{C}_w^B$, we get $\mathcal{T}_w^B$, which consists of $(P, Q) \in \mathcal{T}$ such that

- $[\mathrm{word}(P)]_H = w$.
- ????

How to describe the second condition?

Right keys of decreasing tableaux

We may anti-rectify a decreasing tableau using K-jeu-de-taquin (Kjdt) [Thomas,Yong 2009].

5	4	1
3	2	
2		

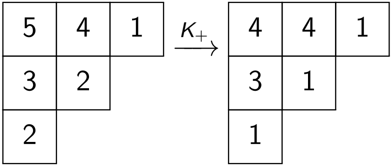
$\rightarrow$

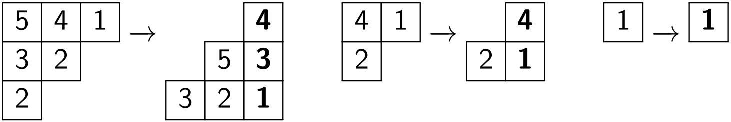
		4
	5	3
3	2	1

Bad News: Anti-rectification is not unique!

Right keys of decreasing tableaux

Good News: [Shimozono, Y 2021] The rightmost column of all anti-rectifications must agree.





Expand $\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$

Recall that $\mathcal{C}_w^B := \{(a,i) \in \mathcal{C} : [a]_H = w, \text{bounded}\}$.
Define
$\mathcal{T}_w^B := \{(P,Q) \in \mathcal{T} : [\text{word}(P)]_H = w, K_+(P) \geq K_-(max(Q))\}$.

Theorem (Shimozono, Y 2021)

Hecke insertion restricts to a bijection from $\mathcal{C}_w^B$ to $\mathcal{T}_w^B$.

$$\begin{aligned}
\mathfrak{G}_w^{(\beta)} &= \sum_{(a,i)\in\mathcal{C}^B_{w^{-1}}} x^{\mathrm{wt}(i)} \beta^{\ell(i)-\ell(w)} \\
&= \sum_{(P,Q)\in\mathcal{T}^B_{w^{-1}}} x^{\mathrm{wt}(Q)} \beta^{\mathrm{ex}(Q)+|\mathrm{shape}(Q)|-\ell(w)} \\
&= \sum_{P} \beta^{|\mathrm{shape}(P)|-\ell(w)} \sum_{Q} x^{\mathrm{wt}(Q)} \beta^{\mathrm{ex}(Q)} \\
&= \sum_{P} \beta^{|\mathrm{shape}(P)|-\ell(w)} \mathfrak{L}^{(\beta)}_{K_+(P)}
\end{aligned}$$

Expand $\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$ example

Let $w = 31524$. Then P can be:

4	3	1
2		

4	3
2	1

4	3	1
2	1	

There right keys are:

3	1	1
1		

3	3
1	1

3	3	1
1	1	

Thus, we have $\mathfrak{G}_w^{(\beta)} = \mathfrak{L}_{301}^{(\beta)} + \mathfrak{L}_{202}^{(\beta)} + \beta\mathfrak{L}_{302}^{(\beta)}$.

Reiner-Yong conjecture

We have

$$\mathfrak{G}_w^{(\beta)} = \sum_{\substack{P \text{ decreasing} \\ [\text{word}(P)]_H = w^{-1}}} \beta^{|\text{shape}(P)| - \ell(w)} \mathfrak{L}_{K_+(P)}^{(\beta)}.$$

Reiner and Yong conjecture:

$$\mathfrak{G}_w^{(\beta)} = \sum_{\substack{P \text{ increasing} \\ [\text{word}(P)]_H = w}} \beta^{|\text{shape}(P)| - \ell(w)} \mathfrak{L}_{K_-(P)}^{(\beta)},$$

where $K_-(\cdot)$ of an increasing tableau is defined analogously.

From decreasing to increasing

Theorem (Shimozono, Y)

There is a bijection from decreasing tableaux to increasing tableaux: $P \to P^\sharp$.
It satisfies:

- $[word(P)]_H = [word(P^\sharp)]_H^{-1}$
- $K_+(P) = K_-(P^\sharp)$

Corollary

The Reiner-Yong conjecture is true.

Thanks for listening!!

- M. Shimozono, and T Yu. Grothendieck to Lascoux expansions. arXiv preprint arXiv:2106.13922 (2021).