Math 182: Hidden Data in Random Matrices

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Today: } PCA

Next: 

Homework 2: Due TONIGHT before midnight.

Office Hours: This week, Kemp's OH

Thursday 9-11a
Friday 11:30a-12:30p
Affine subspace $A \subseteq \mathbb{R}^m$

$A = \mathbf{\mu} + \mathbb{V}$

genuine subspace specified by an orthonormal basis

$\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \ldots, \hat{\mathbf{u}}_d$

translation vector
The core idea of PCA is to find the "best fit affine subspace" for the given data \( x_1, x_2, \ldots, x_n \).

I.e. find a translation vector \( \mu_0 \) and a subspace \( V \) s.t.

\[
x_j - \mu \approx V
\]

Least Squares:

\( M, Q \)
Best Translation Vector

\[ \Phi^d (\mu, Q, \beta_1, \ldots, \beta_N) = \sum_{j=1}^{N} \| y_j - \mu - Q \beta_j \|^2. \]

Find the minimum at critical point (smooth convex function)

Start with \( \mu \). Take the directional derivative

\[ \frac{d}{dt} \Phi^d (\mu + t z_j Q, \beta_1, \ldots, \beta_N) \bigg|_{t=0} \]
Special Case: $m = 2$, $d = 1$

$$x_j = \begin{bmatrix} x_i \ \ y_j \end{bmatrix}$$

$$Q = \hat{u} \in \mathbb{R}^2 \text{ unit vector}$$

$$\beta_j \in \mathbb{R}^1$$

$$x_j \approx \bar{x}_n + \hat{u} \beta_j$$

Assume we've already found $\hat{u}$. 
Actually, the same orthogonality persists in general.

Prop: For each fixed subspace $V$, given by \( \mathbf{Q} \in \mathbb{M}_{m \times d} \),

\[
\min_{\beta_1, \ldots, \beta_n} \sum_{j=1}^{n} \| z_j - \mathbf{Q} \beta_j \|^2 =
\]
\[
\min_{\beta_1, \ldots, \beta_n} \sum_{j=1}^{n} \| \hat{y}_j - Q \beta_j \|^2
\]