Math 182: Hidden Data in Random Matrices

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Today: Probability Review; Gaussians
Next: Unbiased Estimators; MLEs.

HW1: Due Friday, Jan 17 by 11:59 pm (Gradescope)
Lab1: Due Tuesday, Jan 21
Probability: Random Variables

Basic ingredient: probability space \((\Omega, \mathcal{F}, P)\)

A random variable \(X\) is a function which is
Distributions

The distribution of a random variable $X: \Omega \rightarrow \mathbb{R}$ is the probability measure on $\mathbb{R}$ defined by

$$\mu([a,b]) = \mathbb{P}(X \in [a,b])$$

**CDF (Cumulative Distribution Function)**

$$F_X : \mathbb{R} \rightarrow [0,1] \quad F_X(t) = \mathbb{P}(X \leq t)$$

Then $\mathbb{P}(X \in (a,b])$

**Properties:** $F_X$ is non-decreasing, right-continuous

$$\lim_{t \rightarrow -\infty} F_X(t) = 0, \lim_{t \rightarrow +\infty} F_X(t) = 1$$
Jumps in $F_X$ correspond to point masses for $X$:

\[
F_X \quad m \quad \bullet \quad \downarrow \quad \to
\]

$F_X$ is a continuous function iff

Special case: if $F_X$ is differentiable with (nice enough) derivative $F_X' = f_X$, then the Fundamental Theorem of Calculus tells us

In this case, we say $X$ is a random variable, and the function $f_X$ is called its
Most Important Example

Gaussian / Normal density \( N(\mu, \sigma^2) \)

\[
f(t) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}}
\]

If \( Z \sim N(0,1) \)

"standard normal"

then

As with all probability densities, the Gaussian satisfies
Expectation, Moments

If $X$ is a r.v. with density $f_X$, its expectation is

$E(X)$

Change of variables: if $g: \mathbb{R} \to \mathbb{R}$ then

$E(g(X))$

In particular, the moments of $X$ are defined to be

$E(X^n)$

The set of random variables (on $(\Omega, \mathcal{F}, P)$) for which $E(|X|^p) < \infty$ is called $L^p = L^p(\Omega, \mathcal{F}, P)$. By properties of integrals,
Gaussian Moments

Fun calculus exercise: if \( Z \sim N(0,1) \),

\[
E(Z^n) =
\]
Fact: $E : L^1 \rightarrow \mathbb{R}$ is linear.

Variance and Concentration

If $X \in L^2$, its variance is $\text{Var} X = \mathbb{E}((X - \mathbb{E}(X))^2)$.

Its square root $\sqrt{\text{Var}(X)} = \text{SD}(X)$ is standard deviation.

It is a measure of spread / concentration.

Chebyshev's Inequality  
If $X \in L^2$ with $\mathbb{E}(X) = \mu$, $\text{SD}(X) = \sigma$,  

$\forall \ k > 0,$
Random Vectors

A random vector \( \mathbf{X} : \Omega \rightarrow \mathbb{R}^d \) is a vector s.t. \( \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{pmatrix} \). Components are random variables.

A random vector is **jointly continuous** if it has a (multivariate) PDF:

**Caution**: Even if each component of \( \mathbf{X} \) is a continuous r.v., it need not be true that \( \mathbf{X} \) is jointly continuous.

E.g. \( \mathbf{X} = \)
Covariance & Correlation

Given \( x, y \in L^2 \), their covariance is

\[
\text{Cov}(x, y)
\]

Covariance is (almost) an inner product on \( L^2 \).

So we should look at the "angle ratio"

\[
\frac{\text{Cov}(x, y)}{\sqrt{\text{Var}_x} \sqrt{\text{Var}_y}} = \text{Corr}(x, y) = \rho(x, y) \in [-1, 1]
\]

If \( \text{Corr}(x, y) = 0 \), we say the rv's are uncorrelated.
Independence

Two events $A, B \in \Omega$ are (statistically) independent if

$$P(A \cap B) = P(A)P(B).$$

Two random variables $X, Y : \Omega \to \mathbb{R}$ are independent if
Independence and (Joint) Moments

If $\mathbf{X} = [X_1, X_2, \ldots, X_d]^T$ is a random vector, its joint moments are

Proposition: If the components of $\mathbf{X}$ are independent, then the joint moments factorize:

**Pf. (Jointly continuous case)**

$E(X_1^{n_1}X_2^{n_2}\cdots X_d^{n_d}) =$