DSC 155 & MATH 182: “WINTER” 2020
TAKE HOME FINAL EXAM
March 20, 2020

Available | 11:15am | Due | 2:50pm

You are free to work on this exam from 11:30am through 2:30pm. Turn in the exam on Gradescope by 2:50pm.

Please do not collaborate with other humans, and please do not use any online resources—beyond what is on the course webpage (course notes, homework and midterm solutions, and the slides from lectures). You may use your own notes and the approved textbooks for the course. You may also refer to old Piazza posts for this course; but do not post or answer new ones, other than private posts to Prof. Kemp for clarifying questions.

Academic Integrity Pledge

I am fair to my classmates and instructors by not using any unauthorized aids.
I respect myself and my university by upholding educational and evaluative goals.
I am honest in my representation of myself and of my work.
I accept responsibility for my actions, in accord with academic integrity.
I show that I am trustworthy even when no one is watching.

1. (1 point) Affirm your adherence to the Academic Integrity Pledge by copying the following down and signing your name.

I Excel with Integrity.

2. (5 points) Consider the following matrix:

\[ X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}. \]

Find the Singular Value Decompositions (SVD) of \( X \).

3. Let \( g: \mathbb{R} \rightarrow \mathbb{R} \) be a bounded function, and consider the linear statistic

\[ T_n(x_1, \ldots, x_n) = \frac{1}{N} \sum_{j=1}^{N} g(x_j). \]

Let \( X_1, X_2, \ldots \) be i.i.d. random variables, with common density \( f \).
(a) (3 points) Show that $T_N(X_1, \ldots, X_N)$ is a consistent estimator of the parameter

$$\theta = \int_{\mathbb{R}} g(x) f(x) \, dx.$$  

[Hint: Apply the Weak Law of Large Numbers to $g(X_1), g(X_2), \ldots$.]

(b) (3 points) Is $T_N$ an unbiased estimator? Explain.

4. (5 points) Let $N, m \in \mathbb{N}$, and let $X$ be an $m \times N$ matrix whose entries $[X]_{ij}$ are i.i.d. Rademacher random variables:

$$\mathbb{P}([X]_{ij} = -1) = \mathbb{P}([X]_{ij} = 1) = \frac{1}{2}.$$  

Find the distribution of the random variable

$$\text{Tr}(XX^\top).$$  

[Hint: It isn’t very random.]

5. (5 points) Let $\varrho > 0$, and let $f_\varrho$ be the Marčenko–Pastur density with aspect ratio $\varrho$. Show that

$$\frac{1}{\sqrt{\varrho}} f_\varrho(\sqrt{\varrho} x + 1)$$  

converges, as $\varrho \downarrow 0$, to the standard semicircular density:

$$\lim_{\varrho \downarrow 0} \frac{1}{\sqrt{\varrho}} f_\varrho(\sqrt{\varrho} x + 1) = \frac{\sqrt{4 - x^2}}{2\pi} 1_{[-2,2]}(x).$$  

[This is what happens when we consider our usual $m \times N$ i.i.d. feature matrix $X$, except this time we let $\frac{m}{N} \to 0$. Things will blow up unless we rescale; here we must scale $\frac{1}{\sqrt{mN}} X^\top X$ instead of $\frac{1}{N} X^\top X$. The ratio of these two scaling factors is $\sqrt{N/m} \sim 1/\sqrt{\varrho}$. The result for this scaling yields the semicircle law for the histogram of eigenvalues. That’s why your simulations of Wishart matrices in Lab 3 looked semicircular when $\varrho$ was very small (or very large, by the duality $f_{1/\varrho}(x) = f_\varrho(\varrho x)$).]

6. (5 points) In your own words, explain why the sample covariance matrix being an inconsistent estimator of the true covariance is not a bad thing.

7. (3 points) Briefly describe what you learned in this course, and whether you enjoyed learning it.