1. For $\varrho \geq 1$, find the mean of the Marčenko–Pastur density $f_\varrho$. What can you say about the mean of the Marčenko–Pastur distribution when $\varrho < 1$?

2. Let $Z$ be a random vector in $\mathbb{R}^m$ with standard normal distribution $\mathcal{N}(0, I_m)$. Let $d < m$ and let $U \in \mathbb{M}_{d\times m}$ be a matrix with orthonormal rows. Show that the random vector $X = UZ$ in $\mathbb{R}^d$ has the standard normal distribution $\mathcal{N}(0, I_d)$.

Conversely: suppose we start with $X \sim \mathcal{N}(0, I_d)$, and consider the vector $Z' = U^TX$. Is $Z'$ a standard normal random vector in $\mathbb{R}^m$? If not, what can you say about its covariance (in particular about the eigenvalues of its covariance matrix)?

3. Let $X$ and $Y$ be independent standard normal random variables. Define

$$U = \frac{1}{\sqrt{2}}(X - Y), \quad V = \frac{1}{\sqrt{2}}(X + Y).$$

Show that $U$ and $V$ are also standard normal random variables, and that they are uncorrelated. (They are in fact independent, but you need only show they’re uncorrelated.)
1. \[ f_p(x) = \frac{\sqrt{(x-p_+)(p_+-x)}}{2\pi} \mathds{1}_{[p_-,p_+]}(x) \]

This is the density, for \( p > 1 \). In this range, the mean is:

\[
\mathbb{E} = \int_{p_-}^{p_+} x f_p(x) \, dx = \frac{1}{2\pi} \int_{p_-}^{p_+} \sqrt{(x-p_+)(p_+-x)} \, dx
\]

\[
(x-p_+)(p_+-x) = -x^2 + (p_+ + p_-) x - p_+ p_-
\]

\[
p_\pm = (1 \pm \sqrt{p})^2
\]

\[
p_+ + p_- = 1 + 2\sqrt{p} + p
\]

\[
p_+ + p_- = 1 + 2\sqrt{p} + p
\]

\[
p_+ + p_- = 2(1+p)
\]

\[
p_- p_+ = (1+\sqrt{p})^2 (1-\sqrt{p})^2
\]

\[
p_- p_+ = (1-\sqrt{p})^2
\]

\[
\mathbb{E} = \frac{1}{2\pi} \int_{p_-}^{p_+} \sqrt{4p - (x-(1+p))^2} \, dx
\]

Semicircle, centered at \( 1+p \), with radius \( 2\sqrt{p} \). Total area \( \frac{1}{2} (4p) \pi \),

\[
\mathbb{E} = \frac{1}{2\pi} \cdot \frac{1}{2} \cdot 4p \cdot \pi = p
\]
When \( p < 1 \), the M-P distribution is point mass
\[ f_{p(x)} \to (1-p) S_0 + p. \] density of total mass \( p. \)

\[ \therefore E = (1-p) \cdot 0 + \int_{\cdot}^{\cdot} f_{p(x)} dx \]

\[ = 0 \quad \text{computed last page} = p. \]

\[ \therefore E = p \quad \text{for all} \quad p \geq 1. \]

2. We use the fact (proved in the early course notes) that \( AZ \) is \( N(0, AAT) \). (We'll verify the explicit Gaussianness of a linear transformation of Gaussians in problem 3.)

So we just care about the covariance.

In the first part, \( A = U \) has o.n. rows.

\[ : A^T = U^T U = I. \quad \text{So} \quad X = UZ \sim N(0, I). \]

\[ \text{on columns} \]
For the second part: row \( x \sim N^\prime(0, I_d) \), and
\[
Z' = U^T X
\]
m\times 1 \quad m \times d \quad d \times 1
\[
\therefore Z' \sim N^\prime(0, U(U^T)) = N^\prime(0, U^T U) \quad \text{m\times m}
\]

\[
\text{rank}(U^T U)
\]
\[
\leq \text{max} (\text{rank}(U^T), \text{rank}(U))
\]
\[
= \text{rank}(U) = d \quad \text{b/c} \quad d < m.
\]
\[\uparrow\]
d \quad \text{m\times m} \quad \text{(i.e., m\text{, independent) rows)}
\]

\[
\therefore \text{rank}(U^T U) = d \leq m \quad \text{so} \quad d \leq m.
\]
\[\uparrow\]
\[
\therefore U^T U \neq I.
\]

has 0 as an eigenvalue of
multiplicity m-d.
3. We could just apply \#2.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

\[ U \] 2x2 orthogonal matrix (rotation $\frac{\pi}{4}$ radians ccw)

has or. rows, so \(\checkmark\)

More directly:

To check that \(U, V\) are Gaussian, we could use:

- Convolution densities
- Moment generating function
- Fourier transform
- Stieltjes transform

I'll check this one.
\[ M_u(t) = \mathbb{E}(e^{tX}) = \mathbb{E}(e^{\frac{t}{\sqrt{2}}(X-Y)}) = \mathbb{E}(e^{\frac{t}{\sqrt{2}}X} e^{-\frac{t}{\sqrt{2}}Y}) = \mathbb{E}(e^{\frac{t}{\sqrt{2}}X}) \mathbb{E}(e^{-\frac{t}{\sqrt{2}}Y}) \]

\( X, Y \sim N^0(0,1) \), so \( \mathbb{E}(e^{tX}) = \mathbb{E}(e^{tY}) = e^{\frac{t^2}{2}} \)

\[ M_u(t) = e^{\left(\frac{t}{\sqrt{2}}\right)^2/2} \cdot e^{\left(-\frac{t}{\sqrt{2}}\right)^2/2} = e^{\frac{t^2}{4}} \cdot e^{\frac{t^2}{4}} = e^{\frac{t^2}{2}} \]

\[ \Rightarrow \mathbb{U} \sim N^0(\frac{3}{2},1) \]

\[ \text{Also } \quad M_v(t) = \mathbb{E}(e^{\frac{t}{\sqrt{2}}(X+Y)}) = \mathbb{E}(e^{\frac{t}{\sqrt{2}}X}) \mathbb{E}(e^{\frac{t}{\sqrt{2}}Y}) = e^{\frac{t^2}{4}} \cdot e^{\frac{t^2}{4}} = e^{\frac{t^2}{2}} \]

\[ \Rightarrow \mathbb{V} \sim N^0(\frac{3}{2},1) \]

\[ \text{Finally: } \quad \mathbb{E}(UV) = \mathbb{E}(\frac{1}{\sqrt{2}}(X-Y) \cdot \frac{1}{\sqrt{2}}(X+Y)) = \frac{1}{2} \mathbb{E}(X^2-Y^2) = \frac{1}{2}(\mathbb{E}(X^2) - \mathbb{E}(Y^2)) = \frac{1}{2}(1-1) = 0 \]

\[ \text{and } \quad \mathbb{E}(U) = \mathbb{E}(V) = \frac{1}{\sqrt{2}}(\mathbb{E}(X) \pm \mathbb{E}(Y)) = 0 \]