Abstract

Quantile regression is a powerful tool for learning the relationship between a scalar response and a multivariate predictor in the presence of heavier tails and/or data heterogeneity. In the present paper, we consider statistical inference for quantile regression with large-scale data in the increasing dimension regime. We provide a comprehensive study of a convolution-type smoothing approach to achieving an adequate approximation to computation and inference for quantile regression. The ensuing estimator, which we refer to as conquer, turns the non-differentiable quantile loss function into a twice-differentiable, globally convex, and locally strongly convex surrogate, which admits a fast and scalable Barzilai-Borwein gradient-based algorithm to perform optimization, and a Rademacher multiplier bootstrap method for statistical inference. In the theoretical investigations of the conquer estimator, we establish nonasymptotic error bounds on the Bahadur-Kiefer linearization, from which we show that the asymptotic normality of the smoothed quantile regression estimator holds under a weaker requirement on the dimension of the predictors than needed for the exact quantile regression estimator. Our numerical studies confirm the conquer estimator as a practical and reliable approach to large-scale inference for quantile regression.

Keywords: Bahadur-Kiefer representation; convolution; gradient descent; multiplier bootstrap.
1. Introduction

Quantile regression (QR) is a useful statistical tool for modeling and inferring the relationship between a scalar response \( y \) and a \( p \)-dimensional predictor \( \mathbf{x} \) (Koenker and Bassett, 1978). Compared to the least squares regression that focuses on modeling the conditional mean of \( y \) given \( \mathbf{x} \), QR allows modeling of the entire conditional distribution of \( y \) given \( \mathbf{x} \), and thus provides valuable insights into heterogeneity in the relationship between \( \mathbf{x} \) and \( y \). Moreover, quantile regression is robust against outliers and can be performed for skewed or heavy-tailed response distributions without a correct specification of the likelihood. These advantages make QR an appealing method to explore data features that are invisible to the least squares regression. We refer the reader to Koenker (2005) and Koenker et al. (2017) for an extensive overview of QR in terms of methods, theory, computation, and various extensions under complex data structures.

Quantile regression involves a convex optimization problem with a piecewise linear loss function, also known as the check function and pinball loss. One can formulate QR as a linear programming problem, solvable by the interior point method with a computational complexity of \( O(n^{1+\alpha}p^3 \log n) \) with \( \alpha \in (0, 1/2) \), where \( n \) is the sample size and \( p \) is the parametric dimension. When \( n \) is large relative to \( p \), an efficient algorithm based on preprocessing has an improved complexity of \( O\{(np)^{2(1+\alpha)/3}p^3 \log n + np\} \), and thus can be more efficient than solving a least squares problem when \( p \) is small (Portnoy and Koenker, 1997). However, primarily due to the non-differentiability and lack of strong convexity of the loss function, QR remains computationally expensive for large-scale data when both \( n \) and \( p \) are large. We refer to Chapter 5 of Koenker et al. (2017) for an overview of prevailing computational methods for quantile regression, such as simplex-based algorithms (Barrodale and Roberts, 1974; Koenker and d’Orey, 1987), interior point methods (Portnoy and Koenker, 1997), and alternating direction method of multipliers among other first-order proximal methods (Parikh and Boyd, 2014).

We consider conducting large-scale inference for quantile regression under the setting in which \( p \) is large and \( n \) is even larger. Two general principles have been widely used to suit this purpose. The first uses a nonparametric estimate of the asymptotic variance (Gutenbrunner and Jurečková, 1992) that involves the conditional density of the response given the covariates, yet such an estimate can be fairly unstable. Even if the asymptotic variance is well estimated, its approximation accuracy to the finite-sample variance depends on the design matrix and the quantile level. Resampling methods, on the other hand, provide a more reliable approach to inference for QR under a wide variety of settings (Parzen, Wei and Ying, 1994; He and Hu, 2002; Kocherginsky, He and Hu, 2005; Feng, He and Hu, 2011). Inevitably, the resampling approach requires repeatedly computing QR estimates up to thousands of times, and therefore is unduly expensive for large-scale data.

Theoretically, valid statistical inference is often justified by asymptotic normal approximations to QR estimators. The Bahadur-Kiefer representation of the nonlinear QR estimators is essential to this end, as shown in Arcones (1996) and He and Shao (1996). In large-\( p \) (non)asymptotic settings in which the parametric dimension \( p \) may tend to infinity with the sample size, we refer to Welsh (1989), He and Shao (2000), and Pan and Zhou (2020) for normal approximation results of the QR estimators under fixed and random designs. The question of how large \( p \) can be relative to \( n \) to ensure asymptotic normality has been
addressed by those authors. It is now recognized that we may have to pay a price here as compared to $M$-estimators with smooth loss functions that are at least continuously twice differentiable.

To circumvent the non-differentiability of the QR loss function, Horowitz (1998) proposed to smooth the indicator part of the check function via the survival function of a kernel. This smoothing method, which we refer to as Horowitz's smoothing throughout, has been widely used for various QR-related problems (Whang, 2006; Kaplan and Sun, 2017; Galvao and Kato, 2016; Wang, Stefanski and Zhu, 2012; Wu, Ma and Yin, 2015; de Castro et al., 2019; Chen, Liu and Zhang, 2019). However, Horowitz's smoothing gains smoothness at the cost of convexity, which inevitably raises optimization issues. In general, computing a global minimum of a non-convex function is intractable: finding an $\epsilon$-suboptimal point for a $k$-times continuously differentiable function $f : \mathbb{R}^p \rightarrow \mathbb{R}$ requires at least as many as $(1/\epsilon)^{p/k}$ evaluations of the function and its first $k$ derivatives (Nemirovski and Yudin, 1983).

As we shall see from the numerical studies in Section 5, the convergence of gradient-based algorithms can be relatively slow for high and low quantile levels. To address the aforementioned issue, Fernandes, Guerre and Horta (2019) proposed a convolution-type smoothing method that yields a convex and twice differentiable loss function, and studied the asymptotic properties of the smoothed estimator when $p$ is fixed. The proposed convolution-type smoothing method is also related to a smoothed estimating equation estimator proposed by Kaplan and Sun (2017). To distinguish the aforementioned approach from Horowitz's smoothing, we adopt the term conquer for convolution-type smoothed quantile regression.

In this paper, we first provide an in-depth statistical analysis of conquer under various array (non)asymptotic settings in which $p$ increases with $n$. Our results reveal a key feature of the smoothing parameter, often referred to as the bandwidth: the bandwidth adapts to both the sample size $n$ and dimensionality $p$, so as to achieve a tradeoff between statistical accuracy and computational stability. Since the convolution smoothed loss function is globally convex and locally strongly convex, we propose an efficient gradient descent algorithm with the Barzilai-Borwein stepsize and a Huber-type initialization. The proposed algorithm is implemented via RcppArmadillo (Eddelbuettel and Sanderson, 2014) in the R package conquer. We next focus on large-scale statistical inference (hypothesis testing and confidence estimation) with large $p$ and large $n$. We propose a bootstrapped conquer method that has reduced computational complexity when the conquer estimator is used as initialization. Under appropriate restrictions on dimension, we establish the consistency (or concentration), Bahadur representation, asymptotic normality of the conquer estimator as well as the validity of the bootstrap approximation. In the following, we provide more details on the computational and statistical contributions of this paper.

Theoretically, by allowing $p$ to grow with $n$, the ‘complexity’ of the function classes that we come across in the analysis also increases with $n$. Conventional asymptotic tools for proving the bootstrap validity are based on weak convergence arguments (van der Vaart and Wellner, 1996), which are not directly applicable in the finite-sample setting. In this paper we turn to a more refined and self-contained analysis, and prove a new local restricted strong convexity (RSC) property for the empirical smoothed quantile loss. This validates the key merit of convolution-type smoothing, i.e., local strong convexity. The smoothing method involves a bandwidth, denoted by $h$. Theoretically, we show that with sub-exponential random covariates (relaxing the bounded covariates assumption in Fernan-
des, Guerre and Horta (2019)), conquer exhibits an $\ell_2$-error in the order of $\sqrt{(p + t)/n + h^2}$ with probability at least $1 - 2e^{-t}$. When $h$ is of order $\{(p + t)/n\}^\gamma$ for any $\gamma \in [1/4, 1/2]$, the conquer estimation is first-order equivalent to QR. Under slightly more stringent sub-Gaussian condition on the covariates, we show that the Bahadur-Kiefer linearization error of conquer is of order $(p + t)/(nh^{1/2}) + h\sqrt{(p + t)/n + h^3}$ with probability at least $1 - 3e^{-t}$. Based on such a representation, we establish a Berry-Esseen bound for linear functionals of conquer, which lays the theoretical foundation for testing general linear hypotheses, encompassing covariate-effect analysis, analysis of variance, and model comparisons, to name a few. It is worth noting that with a properly chosen $h$, the linear functional of conquer is asymptotically normal as long as $p^{8/3}/n \to 0$, which improves the best known growth condition on $p$ for standard QR (Welsh, 1989; He and Shao, 2000; Pan and Zhou, 2020). We attribute this gain to the effect of smoothing. Under similar conditions, we further establish upper bounds on both estimation and Bahadur-Kiefer linearization errors for the bootstrapped conquer estimator.

In the context of nonparametric density or regression estimation, it is known that when higher-order kernels are used (and if the density or regression function has enough derivatives), the bias is proportional to $h^\nu$ for some $\nu \geq 4$ which is of better order than $h^2$. While a higher-order kernel has negative parts, the resulting smoothed loss is non-convex and thus brings the computational issue once again. Motivated by the two-stage procedure proposed by Bickel (1975) whose original idea is to improve an initial estimator that is already consistent but not efficient, we further propose a one-step conquer estimator using higher-order kernels but without the need for solving a large-scale non-convex optimization. With increasing degrees of smoothness, the one-step conquer is asymptotically normal under a milder dimension constraint of roughly $p^2/n \to 0$.

To better appreciate the computational feasibility of conquer for large-scale problems, we compare it with standard QR on large synthetic datasets, where the latter is implemented by the R package quantreg (Koenker, 2019) using the Frisch-Newton approach after preprocessing "pfn". We generate independent data vectors $\{y_i, x_i\}_{i=1}^n$ from a linear model $y_i = \beta_0^* + \langle x_i, \beta^* \rangle + \varepsilon_i$, where $(\beta_0^*, \beta^*)^T = (1, \ldots, 1)^T \in \mathbb{R}^{p+1}$, $x_i \sim N_p(0, I)$, and the independent errors $\varepsilon_i \sim t_2$, for $i = 1, 2, \ldots, n$. We report the estimation error and elapsed time for increasing sample sizes $n \in \{1000, 5000, 10000, \ldots, 100000\}$ and dimension $p = [n^{1/2}]$, the largest integer that is less than or equal to $n^{1/2}$. Figure 1 displays the average estimation error and average elapsed time based on 20 Monte Carlo samples. This experiment shows promise of conquer as a practically useful tool for large-scale quantile regression analysis. More empirical evidence will be given in the latter section.

The rest of the paper is organized as follows. We start with a brief review of linear quantile regression and the convolution-type smoothing method in Section 2. Explicit forms of the smoothed check functions are provided for several representative kernel functions in nonparametric statistics. We introduce the multiplier bootstrap for statistical inference in Section 2.4. In Section 3, we provide a comprehensive theoretical study of conquer from a nonasymptotic viewpoint, which directly leads to array asymptotic results. Specifically, the bias incurred by smoothing the quantile loss is characterized in Section 3.1. In Section 3.2, we establish the rate of convergence, Bahadur-Kiefer representation, and Berry-Esseen bound for conquer in a large-$p$ and larger-$n$ regime. Results for its bootstrap counterpart are provided in Section 3.3. A Barzilai-Borwein gradient-based algorithm with
Figure 1: A numerical comparison between conquer and QR. The latter is implemented by the R package quantreg using the “pfn” method. Panels (a) and (b) display, respectively, the “estimation error versus sample size” and “elapsed time versus sample size” as the size of the problem increases.

a Huber-type warm start is detailed in Section 4. We conclude the paper with an extensive numerical study in Section 5 to illustrate the finite-sample performance of conquer in large-scale quantile regression analysis. We defer the proofs of all theoretical results as well as the full details of the one-step conquer to online supplementary materials.

NOTATION: For every integer $k \geq 1$, we use $\mathbb{R}^k$ to denote the $k$-dimensional Euclidean space. The inner product of any two vectors $\mathbf{u} = (u_1, \ldots, u_k)^T, \mathbf{v} = (v_1, \ldots, v_k)^T \in \mathbb{R}^k$ is defined by $\mathbf{u}^T \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^k u_i v_i$. We use $\| \cdot \|_p$ ($1 \leq p \leq \infty$) to denote the $\ell_p$-norm in $\mathbb{R}^k$: $\| \mathbf{u} \|_p = (\sum_{i=1}^k |u_i|^p)^{1/p}$ and $\| \mathbf{u} \|_\infty = \max_{1 \leq i \leq k} |u_i|$. Throughout this paper, we use bold capital letters to represent matrices. For $k \geq 2$, $\mathbf{I}_k$ represents the identity matrix of size $k$. For any $k \times k$ symmetric matrix $\mathbf{A} \in \mathbb{R}^{k \times k}$, $\| \mathbf{A} \|_2$ denotes the operator norm of $\mathbf{A}$. If $\mathbf{A}$ is positive semidefinite, we use $\| \cdot \|_\mathbf{A}$ to denote the vector norm linked to $\mathbf{A}$ given by $\| \mathbf{u} \|_\mathbf{A} = \| \mathbf{A}^{1/2} \mathbf{u} \|_2, \mathbf{u} \in \mathbb{R}^k$. For $r \geq 0$, define the Euclidean ball and sphere in $\mathbb{R}^k$ as $\mathbb{B}^k(r) = \{ \mathbf{u} \in \mathbb{R}^k : \| \mathbf{u} \|_2 \leq r \}$ and $\mathbb{S}^{k-1}(r) = \partial \mathbb{B}^k(r) = \{ \mathbf{u} \in \mathbb{R}^k : \| \mathbf{u} \|_2 = r \}$, respectively. For two sequences of non-negative numbers $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$, $a_n \lesssim b_n$ indicates that there exists a constant $C > 0$ independent of $n$ such that $a_n \leq C b_n$; $a_n \gtrsim b_n$ is equivalent to $b_n \lesssim a_n$; $a_n \asymp b_n$ is equivalent to $a_n \lesssim b_n$ and $b_n \lesssim a_n$.

2. Smoothed Quantile Regression

2.1 The linear quantile regression model

Given a univariate response variable $y \in \mathbb{R}$ and a $p$-dimensional covariate vector $\mathbf{x} = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ with $x_1 \equiv 1$, the primary goal here is to learn the effect of $\mathbf{x}$ on the distribution of $y$. Let $F_{y|x}(\cdot)$ be the conditional distribution function of $y$ given $\mathbf{x}$. The
dependence between $y$ and $x$ is then fully characterized by the conditional quantile functions of $y$ given $x$, denoted as $F_{y|x}^{-1}(\tau)$, for $0 < \tau < 1$. We consider a linear quantile regression model at a given $\tau \in (0, 1)$, that is, the $\tau$-th conditional quantile function is

$$F_{y|x}^{-1}(\tau) = \langle x, \beta^*(\tau) \rangle,$$

where $\beta^*(\tau) = (\beta_1^*(\tau), \ldots, \beta_p^*(\tau))^T \in \mathbb{R}^p$ is the true quantile regression coefficient.

Let $\{(y_i, x_i)\}_{i=1}^n$ be a random sample from $(y, x)$. The standard quantile regression estimator (Koenker and Bassett, 1978) is then given as

$$\hat{\beta}^{\tau} \in \min_{\beta \in \mathbb{R}^p} \hat{Q}(\beta) = \min_{\beta \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^n \rho_\tau(y_i - \langle x_i, \beta \rangle),$$

where

$$\rho_\tau(u) = u\{\tau - 1(u < 0)\}$$

is the $\tau$-quantile loss function, also referred to as the pinball loss and check function. Statistical properties of $\hat{\beta}(\tau)$ have been extensively studied. We refer the reader to Koenker (2005) and Koenker et al. (2017) for more details.

### 2.2 Smoothed estimation equation and convolution-type smoothing

Let $Q(\beta) = \mathbb{E}\{\hat{Q}(\beta)\}$ be the population quantile loss function. Under mild conditions, $Q(\cdot)$ is twice differentiable and strongly convex in a neighborhood of $\beta^*$ with Hessian matrix $D := \nabla^2 Q(\beta^*) = \mathbb{E}\{f_{\varepsilon|x}(0)xx^T\}$, where $f_{\varepsilon|x}(\cdot)$ denotes the conditional density of $\varepsilon$ given $x$. In contrast, the empirical quantile loss $\hat{Q}(\cdot)$ is not differentiable at $\beta^*$, and its “curvature energy” is concentrated at a single point. This is substantially different from other widely used loss functions that are at least locally strongly convex, such as the squared or logistic loss. The non-smoothness property not only brings challenge to theoretical analysis, but more importantly, also prevents gradient-based optimization methods from being efficient.

In his seminal work, Horowitz (1998) proposed to directly smooth the check function $\rho_\tau(\cdot)$ to obtain $\rho_{\tau,Horo}(u) = u\{\tau - G(-u/h)\}$, where $G(\cdot)$ is a smooth function and $h > 0$ is the smoothing parameter or bandwidth. However, Horowitz’s smoothing gains smoothness at the cost of convexity, which inevitably raises optimization issues especially when $p$ is large.

On the other hand, by the first-order condition, the population parameter $\beta^*$ satisfies the moment condition

$$\nabla Q(\beta^*) = \mathbb{E}\left[\left\{1(y < x^T\beta) - \tau\right\}x\right]_{\beta = \beta^*} = 0.$$

This property motivates a smoothed estimating equation (SEE) estimator (Whang, 2006; Kaplan and Sun, 2017), defined as the solution to the smoothed moment condition

$$\frac{1}{n} \sum_{i=1}^n \left\{G((x_i, \beta) - y_i)/h) - \tau\right\}x_i = 0.$$
From an M-estimation viewpoint, the aforementioned SEE estimator can be equivalently defined as a minimizer of the empirical smoothed loss function

$$
\hat{Q}_h(\beta) = \frac{1}{n} \sum_{i=1}^{n} \ell_h(y_i - \langle x_i, \beta \rangle) \quad \text{with} \quad \ell_h(u) = (\rho_r * K_h)(u) = \int_{-\infty}^{\infty} \rho_r(v) K_h(v - u) \, dv,
$$

(5)

where $K(\cdot)$ is a kernel function, $K_h(u) = (1/h)K(u/h)$, and $*$ denotes the convolution operator. Therefore, as stated in the Introduction, we refer to the aforementioned smoothing method as conquer throughout the paper. The ensuing conquer estimator is given by

$$
\hat{\beta}_h = \hat{\beta}_h(\tau) \in \text{argmin}_{\beta \in \mathbb{R}^p} \hat{Q}_h(\beta).
$$

(6)

As we shall see later, the ideal choice of bandwidth should adapt to the sample size $n$ and dimension $p$, while the quantile level $\tau$ is prespecified and fixed. Thus, the dependence of $\hat{\beta}_h$ and $\hat{Q}_h(\cdot)$ on $\tau$ will be assumed without display. Commonly used kernel functions include: (a) uniform kernel $K(u) = (1/2)1(|u| \leq 1)$, (b) Gaussian kernel $K(u) = \phi(u) := (2\pi)^{-1/2}e^{-u^2/2}$, (c) logistic kernel $K(u) = e^{-u}/(1+e^{-u})^2$, (d) Epanechnikov kernel $K(u) = (3/4)(1-u^2)1(|u| \leq 1)$, and (e) triangular kernel $K(u) = (1-|u|)1(|u| \leq 1)$. Explicit expressions of the corresponding smoothed loss function $\rho_r * K_h$ will be given in Section 4.

The convolution-type kernel smoothing yields an objective function $\beta \mapsto \hat{Q}_h(\beta)$ that is twice continuously differentiable with gradient and hessian matrix

$$
\nabla \hat{Q}_h(\beta) = \frac{1}{n} \sum_{i=1}^{n} \{ \bar{K}(\langle x_i^T \beta - y_i \rangle/h) - \tau \} x_i \quad \text{and} \quad \nabla^2 \hat{Q}_h(\beta) = \frac{1}{n} \sum_{i=1}^{n} K_h(y_i - \langle x_i^T \beta \rangle x_i) x_i x_i^T,
$$

(7)

respectively, where $\bar{K}(u) := \int_{-\infty}^{u} K(t) \, dt$. Provided that $K$ is non-negative, $\hat{Q}_h(\cdot)$ is a convex function for any $h > 0$, and $\hat{\beta}_h = \hat{\beta}_h(\tau)$ satisfies the first-order condition $\nabla \hat{Q}_h(\hat{\beta}_h) = 0$. This reveals the connection between SEE and conquer methods. Together, the smoothness and convexity of $\hat{Q}_h(\cdot)$ warrant the superior computation efficiency of first-order gradient based algorithms for solving large-scale smoothed quantile regressions. The computational aspect of conquer will be discussed in Section 4.

When the dimension $p$ is fixed, asymptotic properties of the SEE or conquer estimator have been studied by Kaplan and Sun (2017) and Fernandes, Guerre and Horta (2019). The former used a higher-order kernel to deal with the instrumental variables QR problem, and the latter showed that the conquer estimator has a lower asymptotic mean squared error than Horowitz’s smoothed estimator, and also has a smaller Bahadur linearization error than the standard QR in the almost sure sense. The optimal order of the bandwidth based on the asymptotic mean squared error is unveiled as a function of $n$. In Section 3, we will establish exponential concentration inequalities and nonasymptotic Bahadur representation for the conquer estimator, while allowing the dimension $p$ to grow with the sample size $n$. Our results reveal a key feature of the smoothing parameter: the bandwidth should adapt to both the sample size $n$ and dimensionality $p$, so as to achieve a tradeoff between statistical accuracy and computational stability.
2.3 Connection to related existing work

The idea of smoothing the piecewise linear loss, to the best of our knowledge, dates back to Amemiya (1982) in the context of median regression. Specifically, Amemiya (1982) proposed a smoothed approximation of the absolute value function that has the form

\[ u \mapsto u + 2h \log(1 + e^{-u/h}), \quad u \in \mathbb{R}, \]

where \( h = n^{-\gamma} \) with \( \gamma \in (1/3, 1/2) \). This smoothing device, however, is mainly used therein to simplify the analysis of the asymptotic behavior of a two-stage median regression estimator.

Horowitz (1998) proposed a kernel smoothing method of similar type for bootstrapping the median regression estimator. The idea is to replace the indicator function in \( \rho_\tau(u) = u\{\tau - 1(u < 0)\} \) with a smoothed counterpart, leading to

\[ \ell_{\text{Hor}}(u) = u\{\tau - K(-u/h)\}, \quad (8) \]

where \( K(u) = \int_{-\infty}^u K(t) \, dt \), and \( K(\cdot) \) is a symmetric kernel function. The key difference between the conquer loss (5) and Horowitz’s loss (8) is that the former is globally convex, while Horowitz’s loss is not. This is illustrated in Figure 2. We refer to Fernandes, Guerre and Horta (2019) for an in-depth comparison between convolution smoothing and Horowitz’s smoothing in terms of asymptotic mean squared errors of the smoothed estimators.

A closer inspection reveals that the smoothed function considered by Amemiya (1982) is a convolution-type smoothed loss with a logistic kernel. More recently, Yi and Huang (2017) considered a smoothing approximation for the quantile loss, which they refer to as the Huber approximation. Their goal was to compute the regularized QR estimator but via a smoothed optimization method. The corresponding loss function also falls into the general framework of convolution-type smoothing, with \( K \) taken as the uniform kernel. The general statistical theory developed in this paper applies to those special cases.

2.4 Multiplier bootstrap inference

In this section, we consider a multiplier bootstrap procedure to construct confidence intervals for conquer. Independent of the observed sample \( X_n = \{(y_i, x_i)\}_{i=1}^n \), let \( \{w_i\}_{i=1}^n \) be independent and identically distributed random variables with \( E(w_i) = 1 \) and \( \text{var}(w_i) = 1 \).

Recall that \( \hat{\beta}_h = \hat{\beta}_h(\tau) = \min_{\beta \in \mathbb{R}^p} Q_h(\beta) \) is the conquer estimator. If the minimizer is not unique, we take any of the minima as \( \hat{\beta}_h = (\hat{\beta}_{h,1}, \ldots, \hat{\beta}_{h,p})^T \).

The proposed bootstrap method, which dates back to Dudewicz (1992) and Barbe and Berthelot (1995), is based on reweighting the summands of \( Q_h(\cdot) \) with random weights \( w_i \). The resulting weighted quantile loss \( \hat{Q}_h \) is

\[ \hat{Q}_h^\beta = \frac{1}{n} \sum_{i=1}^n w_i \ell_h(y_i - \langle x_i, \beta \rangle), \quad (9) \]

where \( \ell_h(u) = (\rho_\tau * K_h)(u) \) is as in (5). We refer to Chatterjee and Bose (2005) for a general asymptotic theory for weighted bootstrap for estimating equations, where a class of bootstrap weights is considered. Extensions to semiparametric \( M \)-estimation can be found in Ma and Kosorok (2005) and Cheng and Huang (2010).
Let \( \mathbb{E}^* \) and \( \mathbb{P}^* \) be the conditional expectation and probability given the observed data \( X_n \), respectively. Observe that \( \mathbb{E}^* \{ \hat{Q}_h^*(\beta) \} = \hat{Q}_h(\beta) \) for any \( \beta \in \mathbb{R}^p \). Consequently, we have

\[
\operatorname{argmin}_{\beta \in \mathbb{R}^p} \mathbb{E}^* \{ \hat{Q}_h^*(\beta) \} = \operatorname{argmin}_{\beta \in \mathbb{R}^p} \hat{Q}_h(\beta) = \hat{\beta}_h.
\]

This simple and yet important observation motivates the following multiplier bootstrap statistic:

\[
\hat{\beta}_h^b = \hat{\beta}_h(\tau) \in \operatorname{argmin}_{\beta \in \mathbb{R}^p} \hat{Q}_h(\beta).
\]

(10)

To retain convexity of the loss function, non-negative random weights are preferred, such as \( w_i \sim \text{Exp}(1) \), i.e., exponential distribution with rate 1, and \( w_i = 1 + e_i \), where \( e_i \) are independent Rademacher random variables. We can construct confidence intervals based on the bootstrap estimates using one of the three classical methods, the percentile method, the pivotal method, and the normal-based method. To be specific, for each \( q \in (0, 1) \) and \( 1 \leq j \leq p \), define the (conditional) \( q \)-quantile of \( \hat{\beta}_{h,j} \)—the \( j \)th coordinate of \( \hat{\beta}_h \in \mathbb{R}^p \)—given the observed data as \( c_{j}^b(q) = \inf \{ t \in \mathbb{R} : \mathbb{P}^*(\hat{\beta}_{h,j} \leq t) \geq q \} \). Then, for a prespecified nominal level \( \alpha \in (0, 1) \), the corresponding \( 1 - \alpha \) bootstrap percentile and pivotal confidence intervals (CIs) for \( \beta_{j}^* \) (\( j = 1, \ldots, p \)) are, respectively,

\[
\left[ c_{j}^b(\alpha/2), c_{j}^b(1 - \alpha/2) \right] \quad \text{and} \quad \left[ 2\hat{\beta}_{h,j} - c_{j}^b(1 - \alpha/2), 2\hat{\beta}_{h,j} - c_{j}^b(\alpha/2) \right].
\]

Numerically, \( c_{j}^b(q) \) (\( q \in \{ \alpha, 1 - \alpha/2 \} \)) can be calculated with any specified precision by the simulation. In the \texttt{R} package \texttt{conquer}, the default number of bootstrap replications is set as \( B = 1000 \).
In the next section, we will present a finite-sample theoretical framework for convolution-type smoothed quantile regression, including the concentration inequality and nonasymptotic Bahadur representation for both the conquer estimator (6) and its bootstrap counterpart (10) using Rademacher multipliers. As a by-product, a Berry-Esseen-type inequality (see Theorem 3) states that, under certain constraints on the (growing) dimensionality and bandwidth, the distribution of any linear projection of $\hat{\beta}_h$ converges to a normal distribution as the sample size increases to infinity. Informally, for any given deterministic vector $a \in \mathbb{R}^p$, the scaled statistic $n^{1/2}(a, \hat{\beta}_h - \beta^*)$ is asymptotically normally distributed with asymptotic variance $\sigma_n^2 := \tau(1 - \tau) a^T D^{-1} \Sigma D^{-1} a$. To make inference based on such asymptotic results, we need to consistently estimate the asymptotic variance. Fernandes, Guerre and Horta (2019) suggested the following estimators

$$\hat{D}_h := \nabla^2 \hat{Q}_h(\hat{\beta}_h) = \frac{1}{n} \sum_{i=1}^n K(\hat{\varepsilon}_i/h) \cdot xx_i^T$$
and
$$\hat{V}_h := \frac{1}{nh} \sum_{i=1}^n \{(K(-\hat{\varepsilon}_i/h) - \tau) \}^2 x_i x_i^T$$

of $D$ and $\tau(1 - \tau) \Sigma$, respectively, where $\hat{\varepsilon}_i = y_i - \langle x_i, \hat{\beta}_h \rangle$ are fitted residuals. The ensuing 1-α normal-based CIs are given by $\hat{\beta}_{h,j} \pm \hat{\sigma}_n^{-1} (1 - \alpha/2) \cdot n^{-1/2} (\hat{D}_h^{-1} \hat{V}_h \hat{D}_h^{-1})_{jj}, j = 1, \ldots, p$. However, the normal approximations to the CI suffer from the sensitivity to the smoothing needed to estimated the conditional densities, namely, the matrix $D = \mathbb{E}\{f_{\varepsilon|x}(0)x x^T\}$. This is especially true when $\tau$ is in the upper or lower tail. See Section 5.3 for a numerical comparison between normal approximation and bootstrap calibration for confidence construction at high and low quantile levels. As we shall see, the normal-based CIs can be exceedingly wide and thus inaccurate under these situations.

3. Statistical Analysis

Under the linear quantile regression model in (1), we write, for convenience, the generic data vector $(y, x)$ in a linear model form:

$$y = \langle x, \beta^*(\tau) \rangle + \varepsilon(\tau),$$

where the random variable $\varepsilon(\tau)$ satisfies $\mathbb{P}\{\varepsilon(\tau) \leq 0|x\} = \tau$. Let $f_{\varepsilon|x}(\cdot)$ be the conditional density function of the regression error $\varepsilon = \varepsilon(\tau) \text{ given } x$. We first derive upper bounds for the smoothing bias under mild regularity conditions on the conditional density $f_{\varepsilon|x}$ and the kernel function.

3.1 Smoothing bias

**Condition 3.1** (Kernel function). Let $K(\cdot)$ be a symmetric and non-negative function that integrates to one, that is, $K(u) = K(-u), K(u) \geq 0$ for all $u \in \mathbb{R}$ and $\int_{-\infty}^{\infty} K(u) \, du = 1$. Moreover, $K(\cdot)$ is uniformly bounded with $\kappa_u := \sup_{u \in \mathbb{R}} K(u) < \infty$.

We will use the notation $\kappa_k = \int_{-\infty}^{\infty} |u|^k K(u) \, du$ for $k \geq 1$. Furthermore, we define the population smoothed loss function $Q_h(\beta) = \mathbb{E}\{\hat{Q}_h(\beta)\}$, $\beta \in \mathbb{R}^p$ and the pseudo parameter

$$\beta_h(\tau) \in \arg\min_{\beta \in \mathbb{R}^p} Q_h(\beta),$$

(12)
which is the population minimizer under the smoothed quantile loss. For simplicity, we write \( \beta^* = \beta^*(\tau) \) and \( \beta_h^* = \beta_h^*(\tau) \) hereinafter. In general, \( \beta_h^* \) differs from \( \beta^* \), and we refer to \( \|\beta_h^* - \beta^*\|_2 \) as the approximation error or smoothing bias.

**Condition 3.2** (Conditional density). There exist \( \bar{f} \geq f > 0 \) such that \( f \leq f_{\varepsilon|x}(0) \leq \bar{f} \) almost surely (for all \( x \)). Moreover, there exists a constant \( l_0 > 0 \) such that \( |f_{\varepsilon|x}(u) - f_{\varepsilon|x}(0)| \leq l_0 |u| \) for all \( u \in \mathbb{R} \) almost surely.

**Condition 3.3** (Random design: moments). The (random) vector \( x \in \mathbb{R}^p \) of covariates satisfies \( m_3 := \sup_{u \in \mathbb{R}^{p-1}} \mathbb{E}(\langle (\Sigma^{-1/2}x, u) \rangle^3) < \infty \), where \( \Sigma = \mathbb{E}(xx^\top) \) is positive definite.

If \( x \) is Gaussian, then Condition 3.3 holds trivially. Heavier-tailed distributions of \( x \) is excluded here so that we can expect standard rates of convergence for the quantile regression estimates. The following result characterizes the smoothing bias from a nonasymptotic viewpoint.

**Proposition 1.** Assume Conditions 3.1–3.3 hold. There exist constants \( c_1, c_2 > 0 \) depending on \( (\bar{f}, l_0, m_3, \kappa_1, \kappa_2) \) such that for any \( 0 < h \leq c_1, \beta_h^* \) is the unique minimizer of \( \beta \mapsto Q_h(\beta) \) and satisfies

\[
\|\beta_h^* - \beta^*\|_2 \leq c_2 h^2. \tag{13}
\]

In addition, assume \( f_{\varepsilon|x}(\cdot) \) is continuously differentiable and satisfies almost surely that \( |f'_{\varepsilon|x}(u) - f'_{\varepsilon|x}(0)| \leq l_1 |u| \) for some constant \( l_1 > 0 \). Then

\[
\left\| \mathbf{D}(\beta_h^* - \beta^*) + \frac{1}{2} \kappa_2 h^2 \cdot \mathbb{E}\{f'_{\varepsilon|x}(0)x\} \right\|_{\Omega} \leq c_3 h^3, \tag{14}
\]

where \( \mathbf{D} = \mathbb{E}\{f_{\varepsilon|x}(0)xx^\top\} \), \( \Omega = \Sigma^{-1} \) and \( c_3 > 0 \) is a constant depending on \( (\bar{f}, l_0, l_1, m_3) \) and \( K(\cdot) \).

To better understand the bounds (13) and (14), note that \( \|\beta_h^* - \beta^*\|_2^2 = \mathbb{E}(x, \beta_h^* - \beta^*)^2 \) is the average prediction smoothing error. Interestingly, the upper bound on the right-hand side is dimension-free given \( h \) as long as the uniform third moment \( m_3 \) in Condition 3.3 is dimension-free. Another interesting implication is that, when both \( f_{\varepsilon|x}(0) \) and \( f'_{\varepsilon|x}(0) \) are independent of \( x \), i.e., \( f_{\varepsilon|x}(0) = f_{\varepsilon}(0) \) and \( f'_{\varepsilon|x}(0) = f'_{\varepsilon}(0) \), the leading term in the bias simplifies to

\[
\frac{1}{2} \kappa_2 h^2 \cdot \mathbf{D}^{-1}\mathbb{E}\{f'_{\varepsilon|x}(0)x\} = \frac{f'_{\varepsilon}(0)}{2f_{\varepsilon}(0)} \kappa_2 h^2 \cdot \Sigma^{-1}\mathbb{E}(x) = \frac{f'_{\varepsilon}(0)}{2f_{\varepsilon}(0)} \kappa_2 h^2 \cdot \left[ \frac{1}{0_{p-1}} \right].
\]

In other words, the smoothing bias is concentrated primarily on the intercept. In the asymptotic setting where \( p \) is fixed, and \( h = o(1) \) as \( n \to \infty \), we refer to Theorem 1 in Fernandes, Guerre and Horta (2019) for the expression of asymptotic bias.

### 3.2 Finite sample theory

In this section, we provide two nonasymptotic results, the concentration inequality and the Bahadur-Kiefer representation, for the conquer estimator under random design.
**Condition 3.4** (Random design: sub-exponential case). The predictor $x \in \mathbb{R}^p$ is sub-exponential: there exists $v_0 > 0$ such that $\mathbb{P}\{|\langle u, w \rangle| \geq v_0\|u\|_2t\} \leq 2e^{-t}$ for all $u \in \mathbb{R}^p$ and $t \geq 0$, where $w = \Sigma^{-1/2}x$ with $\Sigma = \mathbb{E}(xx^T)$ is positive-definite.

Condition 3.4 assumes a sub-exponential condition on the random covariates, which encompasses the bounded case considered by Fernandes, Guerre and Horta (2019). For the standardized predictor $w = \Sigma^{-1/2}x$, we define the uniform moment parameters

$$m_k = \sup_{u \in \mathbb{S}^{p-1}} \mathbb{E}|\langle u, w \rangle|^k, \; k = 1, 2, \ldots,$$

with $m_2 = 1$. Under Condition 3.4, a straightforward calculation shows that $m_k \leq 2v_0^k k!$, valid for all $k \geq 1$, where $v_0$ is defined in Condition 3.4. In general, $v_0$ may increase with $p$, but in this paper we are primarily interested in the cases where $v_0$ is a dimension-free constant. This is indeed the case if $x$ is multivariate Gaussian, multivariate Bernoulli, uniform on $[-1,1]^p$, or uniform on the sphere/ball with radius $p^{1/2}$; see Section 3.4 in Vershynin (2018) for more examples.

**Theorem 1.** Assume that Conditions 3.1–3.4 hold with $\kappa_1 = \min_{|u| \leq 1} K(u) > 0$. For any $t > 0$, the smoothed quantile regression estimator $\hat{\beta}_h$ with $\sqrt{(p+t)/n} \lesssim h \lesssim 1$ satisfies

$$\|\hat{\beta}_h - \beta^*\|_\Sigma \leq C(\kappa_1)^{-1} \left(\frac{\sqrt{p+t}}{n} + h^2\right),$$

with probability at least $1 - 2e^{-t}$, where $C > 0$ is a constant depending only on $(v_0, \kappa_2, l_0)$.

The estimation error in (16) is upper bounded by two terms, $h^2$ and $\sqrt{(p+t)/n}$, and can be interpreted as the upper bounds for bias and variance, respectively. The condition $\min_{|u| \leq 1} K(u) > 0$ can be relaxed to $\min_{|u| \leq c} K(u) > 0$ for some $c \in (0,1)$, which will only change the constants encountered in the proof. In particular, for kernels that are compactly supported on $[-1,1]$, we may choose $c = 1/2$ in Theorem 1. Next, we establish a Bahadur representation for conquer. To this end, we impose a slightly more stringent sub-Gaussian condition on the covariates.

**Condition 3.5** (Random design: sub-Gaussian case). The predictor $x \in \mathbb{R}^p$ is sub-Gaussian: there exists $v_1 > 0$ such that $\mathbb{P}\{|\langle u, w \rangle| \geq v_1\|u\|_2t\} \leq 2e^{-t^2/2}$ for all $u \in \mathbb{R}^p$ and $t \geq 0$, where $w = \Sigma^{-1/2}x$.

We are primarily concerned with the cases where $v_1$ is a dimension-free constant that does not depend on $p$; see the remarks made earlier about $v_0$. Assume that the data $\{(y_i, x_i)\}_{i=1}^n$ are generated from the conditional quantile model (1), and write $\varepsilon_i = y_i - (x_i, \beta^*)$ which satisfy $\mathbb{P}(\varepsilon_i \leq 0 | x_i) = \tau$.

**Theorem 2.** Assume Conditions 3.1, 3.2 and 3.5 hold with $\kappa_1 = \min_{|u| \leq 1} K(u) > 0$. Moreover, assume $\sup_{u \in \mathbb{R}} f_\varepsilon(u) \leq f \tilde{f}$ almost surely. Let $t > 0$, and suppose the sample size $n$ and bandwidth $h$ satisfy $\sqrt{(p+t)/n} \lesssim h \lesssim 1$. Then,

$$\left\|D(\hat{\beta}_h - \beta^*) - \frac{1}{n} \sum_{i=1}^n \{\tau - K(\varepsilon_i/h)\} x_i \right\|_{\Omega} \leq C \left(\frac{p+t}{h^{1/2}n} + h \sqrt{\frac{p+t}{n}} + h^3\right),$$

(17)
with probability at least $1 - 3e^{-t}$, where $D = \mathbb{E}\{f_{\epsilon|x}(0)x^Tx\}$, $\Omega = \Sigma^{-1}$, and $C > 0$ is a constant depending only on $(\nu_1, \kappa_2, \kappa_\alpha, \kappa_t, L_0, \delta, \ell)$.

The Bahadur representation can be used to establish the limiting distribution of the estimator or its functionals. Here we consider a fundamental statistical inference problem for testing the linear hypothesis $H_0 : \langle a, \beta^* \rangle = 0$, where $a \in \mathbb{R}^p$ is a deterministic vector that defines a linear functional of interest. It is then natural to consider a test statistic that depends on $n^{1/2}\langle a, \hat{\beta}_h \rangle$. Based on the nonasymptotic result in Theorem 2, we establish a Berry–Esseen bound for the linear projection of conquer.

**Theorem 3.** Assume that the conditions in Theorem 2 hold, and $\sqrt{(p + \log n)/n} \leq h \leq 1$. Then,

$$\Delta_{n,p}(h) := \sup_{x \in \mathbb{R}, a \in \mathbb{R}^p} \left| \mathbb{P}\left\{ \frac{n^{1/2}(a, \hat{\beta}_h - \beta^*)}{\sqrt{\tau(1 - \tau) a^T D^{-1} \Sigma D^{-1} a}} \leq x \right\} - \Phi(x) \right| \leq \frac{p + \log n}{(nh)^{1/2}} + n^{1/2}h^2,$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. In particular, with a bandwidth $h \asymp \{(p + \log n)/n\}^{2/5}$, the normal approximation error $\Delta_{n,p}(h)$ is of order $(p + \log n)^{4/5}n^{-3/10}$.

It is worth noticing that although the bound in (90) holds uniformly over all $a$, the asymptotic variance $\sigma^2 := \tau(1 - \tau) a^T D^{-1} \Sigma D^{-1} a$ depends on $a$.

**Remark 1** (Large-$p$ asymptotics). A broader view of classical asymptotics recognizes that the parametric dimension of appropriate model sequences may tend to infinity with the sample size; that is $p = p_n \to \infty$ as $n \to \infty$. Results with increasing $p$ are available in the context of regularized quantile regression (Belloni and Chernozhukov, 2011; Wang, Wu and Li, 2012; Koenker et al., 2017). In the large-$p$ and larger-$n$ setting with quantile regression estimation without regularization, Welsh (1989) shows that $p^3(\log n)^2/n \to 0$ suffices for a normal approximation, which provides some support to the viability of observed rates of parametric growth in the applied literature. Koenker (1988) considers a sample of 733 wage models appeared in the econometric literature, and finds that $p_n = O(n^{1/4})$ is roughly consistent with empirical practice.

For the (convolution-type) smoothed quantile regression, the Berry–Esseen bound (90) in Theorem 3 immediately yields a large-$p$ asymptotic result. Taking $h = h_n = \{(p + \log n)/n\}^{2/5}$ therein, we find that $n^{1/2}(a, \hat{\beta}_h - \beta^*)$, for any given $a \in \mathbb{R}^p$, is asymptotically normally distributed as long as $p^3/n \to 0$, which improves the best known growth condition on $p$ for quantile regression (Welsh, 1989). For smooth robust regression estimators, asymptotic normality can be proven under less restrictive conditions on $p$. Huber (1973) showed that if the loss is twice differentiable, the asymptotic normality for $(a, \hat{\beta})$, where $a \in \mathbb{R}^p$, holds if $p^3/n \to 0$ as $n$ increases. Portnoy (1985) weakened the condition to $(p \log n)^{3/2}/n \to 0$ if the loss function is four times differentiable and the error distribution is symmetric. For Huber loss that has a Lipschitz continuous derivative, He and Shao (2000) obtained the scaling $p^2 \log p = o(n)$ that ensures the asymptotic normality of arbitrary linear combinations of $\beta$. Table 1 summarizes our discussion here and show that the smoothing for conquer helps ensure asymptotic normality of the estimator under weaker conditions on $p$ than what we need for the usual quantile regression estimator.
Table 1: Summary of scaling conditions required for normal approximation under various loss functions.

<table>
<thead>
<tr>
<th>Loss function</th>
<th>Design</th>
<th>Scaling condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Huber loss (Huber, 1973)</td>
<td>Fixed design</td>
<td>$p^3 = o(n)$</td>
</tr>
<tr>
<td>Four times differentiable loss (Portnoy, 1985)</td>
<td>Fixed design (with symmetric error)</td>
<td>$(p \log n)^{3/2} = o(n)$</td>
</tr>
<tr>
<td>Huber loss (He and Shao, 2000)</td>
<td>Fixed design</td>
<td>$p^2 \log p = o(n)$</td>
</tr>
<tr>
<td>Huber loss (Chen and Zhou, 2020)</td>
<td>Sub-Gaussian</td>
<td>$p^2 = o(n)$</td>
</tr>
<tr>
<td>Quantile loss (Welsh, 1989; He and Shao, 2000)</td>
<td>Fixed design</td>
<td>$p^3 (\log n)^2 = o(n)$</td>
</tr>
<tr>
<td>Quantile loss (Pan and Zhou, 2020)</td>
<td>Sub-Gaussian</td>
<td>$p^3 (\log n)^2 = o(n)$</td>
</tr>
<tr>
<td>Convolution smoothed quantile loss</td>
<td>Sub-Gaussian</td>
<td>$p^{8/3} = o(n)$</td>
</tr>
</tbody>
</table>

Remark 2. In this paper, we show that the accuracy of conquer-based inference via the Bahadur representation (and normal approximations) has an error of rate faster than $n^{-1/4}$ yet slower than $n^{-1/2}$; see Theorems 2 and 3. For standard regression quantiles, Portnoy (2012) proposed an alternative expansion for the quantile process using the “Hungarian” construction of Komlós, Major and Tusnády. This stochastic approximation yields an error of order $n^{-1/2}$ (up to a factor of $\log n$), and hence provides a theoretical justification for accurate approximations for inference in regression quantile models.

3.3 Theoretical guarantees for bootstrap inference

We next investigate the statistical properties of the bootstrapped estimator defined in (10), with a particular focus on the Rademacher multiplier bootstrap. To be specific, we use, in this section and the rest of the paper, the random weights $w_i = 1 + e_i$ for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ are independent Rademacher random variables, that is, $\mathbb{P}(e_i = 1) = \mathbb{P}(e_i = -1) = 1/2$. As before, we consider array (non)asymptotics, so that the bootstrap approximation errors depend explicitly on $(n, p)$.

**Theorem 4.** Assume Conditions 3.1–3.3, and 3.5 hold with $\kappa_l = \min_{|u| \leq 1} K(u) > 0$. Let $t \geq 0$, and suppose the sample size and bandwidth satisfy $\max\{ (p + \log n + t)/n, \sqrt{(p + t)/n} \} \lesssim h \lesssim 1$. Then, there exists some event $\mathcal{E}(t)$ with $\mathbb{P}(\mathcal{E}(t)) \geq 1 - 5e^{-t}$ such that, with $\mathbb{P}^*$-probability at least $1 - 2e^{-t}$ conditioned on $\mathcal{E}(t)$,

$$
\|\hat{\beta}_h^\# - \beta^*\|_\Sigma \leq C \left( \sqrt{\frac{p + t}{n}} + h^2 \right),
$$

where $C > 0$ is a constant depending only on $(v_1, \kappa_2, \kappa_l, l_0, f)$.

Analogously to Theorem 2, we further provide a Bahadur representation result for the bootstrap counterpart $\hat{\beta}_h^\#$, which lays the theoretical foundation for the validity of Rademacher multiplier bootstrap for conquer.

**Theorem 5.** Assume Conditions 3.1, 3.2 and 3.5 hold with $\kappa_l = \min_{|u| \leq 1} K(u) > 0$. Moreover, assume $\sup_{u \in \mathbb{R}} f_{\hat{x}}(u) \leq \hat{f}$ almost surely (in $\mathbf{x}$) and $K(\cdot)$ is $l_K$-Lipschitz continuous. Suppose the sample size satisfies $n \gtrsim q := p + \log n$ and choose the bandwidth $h \asymp (q/n)^{2/5}$. 

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Then, there exists a sequence of events \( \{ \mathcal{F}_n \} \) with \( \mathbb{P}(\mathcal{F}_n) \geq 1 - 8n^{-1} \) such that, with \( \mathbb{P}^* \)-probability at least \( 1 - 3n^{-1} \) conditioned on \( \mathcal{F}_n \),

\[
\left\| \mathbf{D}(\hat{\beta}_h - \beta_h) - \frac{1}{n} \sum_{i=1}^{n} \epsilon_i x_i \{ \tau - K(\epsilon_i/h) \} \right\|_\Omega \lesssim \left( \frac{q}{n} \right)^{4/5} \sqrt{ \left( \frac{q}{n} \right)^{3/5} \left( \frac{p \log n}{n} \right)^{1/4} \sqrt{ \left( \frac{q}{n} \right)^{3/5} \frac{p \log n}{n^{1/2}} } } .
\]

(20)

As suggested by Theorem 3 and Remark 1, if we take the bandwidth \( h \) to be \( \{ (p + \log n)/n \}^{2/5} \), the normal approximation to the conquer estimator is asymptotically accurate provided that \( p^{8/3} = o(n) \) as \( n \to \infty \). For the same \( h \), the right-hand side of (20) is of order \( o(n^{-1/2}) \) provided that \( p^{8/3}(\log n)^{5/3} = o(n) \). Putting these two parts together, we have the following asymptotic bootstrap approximation result. If the dimension \( p = p_n \) satisfies \( p(n \log p)^{5/8} = o(n^{3/8}) \), then, as \( n \to \infty \),

\[
\sup_{x \in \mathbb{R}} |\mathbb{P}(n^{1/2}(a, \hat{\beta}_h - \beta^*) \leq x) - \mathbb{P}^*(n^{1/2}(a, \hat{\beta}_h - \beta^*) \leq x)| \xrightarrow{\mathbb{P}} 0 .
\]

(21)

The proof of (21) follows the same argument as that in the proof of Theorem 3, and therefore is omitted. The additional logarithmic factor in the scaling may be an artifact of the proof technique. For standard quantile regression, Feng, He and Hu (2011) established a fixed-\( p \) asymptotic bootstrap approximation result for wild bootstrap under fixed design.

4. Gradient Descent Methods for Conquer

To solve optimization problems (6) and (10) with non-negative weights, arguably the simplest algorithm is a vanilla gradient descent algorithm (GD). For a prespecified \( \tau \in (0, 1) \) and bandwidth \( h > 0 \), recall that \( \hat{Q}_h(\beta) = (1/n) \sum_{i=1}^{n} \ell_h(y_i - \langle x_i, \beta \rangle) \). Starting with an initial value \( \hat{\beta}^0 \in \mathbb{R}^p \), at iteration \( t = 0, 1, 2, \ldots \), GD computes

\[
\hat{\beta}^{t+1} = \hat{\beta}^t - \eta_t \cdot \nabla \hat{Q}_h(\hat{\beta}^t),
\]

(22)

where \( \eta_t > 0 \) is the stepsize and

\[
\nabla \hat{Q}_h(\beta) = \frac{1}{n} \sum_{i=1}^{n} \left\{ K \left( \frac{y_i - \langle x_i, \beta \rangle}{h} \right) - \tau \right\} x_i \quad \text{with} \quad K(u) = \int_{-\infty}^{u} K(t) \, dt.
\]

In the classical GD method, the stepsize is usually obtained by employing line search techniques. However, line search is computationally intensive for large-scale settings. One of the most important issues in GD is to determine a proper update step \( \eta_t \) decay schedule. A common practice in the literature is to use a diminishing stepsize or a best-tuned fixed stepsize. Neither of these two approaches can be efficient, at least compared to the Newton-Frisch algorithm with preprocessing (Portnoy and Koenker, 1997). Recall that the smoothed loss \( \hat{Q}_h(\cdot) \) is twice differentiable with Hessian

\[
\nabla^2 \hat{Q}_h(\beta) = \frac{1}{nh} \sum_{i=1}^{n} K(\frac{y_i - \langle x_i, \beta \rangle}{h}) x_i x_i^T .
\]
It is therefore intriguing to employ the Newton-Raphson method, which at iteration $t$ would read
\[ \hat{\beta}^{t+1} = \hat{\beta}^t - \{\nabla^2 \hat{Q}_h(\hat{\beta}^t)\}^{-1} \nabla \hat{Q}_h(\hat{\beta}^t). \] (23)

Computing the inverse of the Hessian for large problems can be an expensive operation. Moreover, when $h$ is small, $\nabla^2 \hat{Q}_h(\cdot)$ can have a very large condition number, which leads to slow convergence.

**Remark 3.** Below we list the explicit expressions of the convolution smoothed check functions using several commonly used kernels. Recall that the check function can be written as $\rho_r(u) = |u|/2 + (\tau - 1/2)u$, which, after convolution smoothing, becomes
\[ \ell_h(u) = \frac{1}{2} \int_{-\infty}^{\infty} |u + hv| K(v) \, dv + (\tau - 1/2)u. \]

- (Uniform kernel $K(u) = (1/2)1(|u| \leq 1)$): $\ell_h(u) = (h/2)\ell^U(u/h) + (\tau - 1/2)u$, where $\ell^U(u) := (u^2/2 + 1/2)1(|u| \leq 1) + |u|1(|u| > 1)$ is a Huber-type loss (Huber, 1973).
- (Gaussian kernel $K(u) = (2\pi)^{-1/2}e^{-u^2/2}$): $\ell_h(u) = (h/2)\ell^G(u/h) + (\tau - 1/2)u$, where $\ell^G(u) := (2/\pi)^{1/2}e^{-u^2/2} + u\{1 - 2\Phi(-u)\}$.
- (Logistic kernel $K(u) = e^{-u}/(1 + e^{-u})^2$): $\ell_h(u) = (h/2)\ell^L(u/h) + (\tau - 1/2)u$, where $\ell^L(u) := u + 2\log(1 + e^{-u})$.
- (Epanechnikov kernel $K(u) = (3/4)(1 - u^2)1(|u| \leq 1)$): $\ell_h(u) = (h/2)\ell^E(u/h) + (\tau - 1/2)u$, where $\ell^E(u) := (3u^2/4 - u^4/8 + 3/8)1(|u| \leq 1) + |u|1(|u| > 1)$.
- (Triangular kernel $K(u) = (1 - |u|)1(|u| \leq 1)$): $\ell_h(u) = (h/2)\ell^T(u/h) + (\tau - 1/2)u$, where $\ell^T(u) := (u^2 - |u|3 + 1/3)1(|u| \leq 1) + |u|1(|u| > 1)$.

### 4.1 The Barzilai-Borwein stepsize rule

In this paper, we propose to solve conquer by means of the gradient descent with a Barzilai-Borwein update step (Barzilai and Borwein, 1988), which we refer to as the GD-BB algorithm. The BB method, which is motivated by quasi-Newton methods, has been proven to be very successful in solving nonlinear optimization problems.

Recall the Newton-Raphson method (23), which can be computationally expensive and needs modifications if the Hessian is ill-conditioned especially when $h$ is too small. For this reason, many quasi-Newton methods seek a simple approximation of the inverse Hessian matrix, say $(H^t)^{-1}$, which satisfies the secant equation $H^t\delta^t = g^t$, where for $t = 1, 2, \ldots$,
\[ \delta^t = \hat{\beta}^t - \hat{\beta}^{t-1} \]
\[ g^t = \nabla \hat{Q}_h(\hat{\beta}^t) - \nabla \hat{Q}_h(\hat{\beta}^{t-1}). \] (24)

To relieve the computational bottleneck of inverting a large matrix, the BB method chooses $\eta$ so that $\eta\nabla \hat{Q}_h(\hat{\beta}^t) = (\eta^{-1}I_n)^{-1}\nabla \hat{Q}_h(\hat{\beta}^t)$ “approximates” $(H^t)^{-1}\nabla \hat{Q}_h(\hat{\beta}^t)$. Since $H^t$ satisfies $H^t\delta^t = g^t$, it is more practical to choose $\eta$ such that $(1/\eta)\delta^t \approx g^t$ or $\delta^t \approx \eta g^t$. 

He, Pan, Tan, Zhou
Algorithm 1 Gradient descent with Barzilai-Borwein stepsizes (GD-BB) for solving conquer.

**Input:** data vectors \{ (y_i, x_i) \}_{i=1}^{n}, \tau \in (0, 1), bandwidth \( h \in (0, 1) \), initialization \( \hat{\beta}^{(0)} \), and gradient tolerance \( \delta \).

1. Compute \( \hat{\beta}^{1} = \hat{\beta}^{0} - \nabla \hat{Q}_h(\hat{\beta}^{0}) \)
2. for \( t = 1, 2, \ldots \) do
3. \( \delta^{t} \leftarrow \hat{\beta}^{t} - \hat{\beta}^{t-1}, g^{t} \leftarrow \nabla \hat{Q}_h(\hat{\beta}^{t}) - \nabla \hat{Q}_h(\hat{\beta}^{t-1}) \)
4. \( \eta_{1,t} \leftarrow \langle \delta^{t}, \delta^{t} \rangle / \langle \delta^{t}, g^{t} \rangle, \eta_{2,t} \leftarrow \langle \delta^{t}, g^{t} \rangle / \langle g^{t}, g^{t} \rangle \)
5. \( \eta \leftarrow \min\{ \eta_{1,t}, \eta_{2,t}, 100 \} \) if \( \eta_{1,t} > 0 \) and \( \eta \leftarrow 1 \) otherwise
6. \( \hat{\beta}^{t+1} \leftarrow \hat{\beta}^{t} - \eta \nabla \hat{Q}_h(\hat{\beta}^{t}) \)
7. end for when \( \| \nabla \hat{Q}_h(\hat{\beta}^{t}) \|_2 \leq \delta \)

Via least squares approximations, one may use \( \eta_{1,t}^{-1} = \arg\min_{\alpha} \| \alpha \delta^{t} - g^{t} \|_2^2 \) or \( \eta_{2,t} = \arg\min_{\eta} \| \delta^{t} - \eta g^{t} \|_2^2 \). The BB stepsizes are then defined as

\[
\eta_{1,t} = \frac{\langle \delta^{t}, \delta^{t} \rangle}{\langle \delta^{t}, g^{t} \rangle} \quad \text{and} \quad \eta_{2,t} = \frac{\langle \delta^{t}, g^{t} \rangle}{\langle g^{t}, g^{t} \rangle}.
\]

Consequently, the BB iteration takes the form

\[
\hat{\beta}^{t+1} = \hat{\beta}^{t} - \eta_{\ell,t} \nabla \hat{Q}_h(\hat{\beta}^{t}), \quad \ell = 1 \text{ or } 2.
\]

Note that the BB step starts at iteration 1, while at iteration 0, we compute \( \hat{\beta}^{1} \) using the standard gradient descent with an initial estimate \( \hat{\beta}^{0} \). The procedure is summarized in Algorithm 1. Based on extensive numerical studies, we find that at a fixed \( \tau \), the number of iterations is insensitive to varying \((n, p)\) combinations. Moreover, as \( h \) increases, the number of iterations declines because the loss function is “more convex” for larger \( h \). In Algorithm 1, the quantity \( \delta > 0 \) is called the gradient tolerance, ensuring that the obtained estimate, say \( \hat{\beta} \), satisfies \( \| \nabla \hat{Q}_h(\hat{\beta}) \|_2 \leq \delta \). Provided that \( \delta \lesssim \sqrt{p/n} \), the statistical theory developed in Section 3 prevails. In our R package conquer, we set \( \delta = 10^{-4} \) as the default value; this value can also be specified by the user.

As \( \tau \) approaches 0 or 1, the Hessian matrix becomes ill-conditioned. As a result, the stepsizes computed in GD-BB may sometimes vibrate drastically, causing instability of the algorithm. Therefore, in practice, we set a lower bound for the stepsizes by taking \( \eta_t = \min\{ \eta_{1,t}, \eta_{2,t}, 100 \} \), for \( t = 1, 2, \ldots \). Another cause of an ill-conditioned Hessian arises when we have covariates with very different scales. In this case, the stepsize should be different for each covariate, and a constant stepsize will be either too small or too large for one or more covariates, which correspond to slow convergence. To address this issue, we scale the covariate inputs to have zero mean and unit variance before applying gradient descent.

### 4.2 Warm start via Huber regression

A good initialization helps reduce the number of iterations for GD, and hence facilitates fast convergence. Recall from Remark 3 that with a uniform kernel, the smoothed check function is proximal to a Huber loss (Huber, 1973). Motivated by this subtle proximity, we
We summarize the details in Algorithm 2.

\[ H \] Note that by \( \gamma \) \( \geq \) applying gradient descent with BB update step. \( M \) propose using the Huber \( M \)-estimator as an initial estimate, and then proceed by iteratively applying gradient descent with BB update step.

Let \( H_\gamma(u) = (u^2/2) \mathbb{1}(|u| \leq \gamma) + \gamma(|u| - \gamma/2) \mathbb{1}(|u| > \gamma) \) be the Huber loss parametrized by \( \gamma > 0 \). The Huber \( M \)-estimator is then defined as

\[ \tilde{\beta}_\gamma \in \arg\min_{\beta \in \mathbb{R}^p} \tilde{H}_\gamma(\beta), \text{ where } \tilde{H}_\gamma(\beta) = \frac{1}{n} \sum_{i=1}^{n} H_\gamma(y_i - \langle x_i, \beta \rangle). \] \hspace{1cm} (27)

Note that \( H_1(u) = \ell^U(u) - 1/2 \) for \( \ell^U(\cdot) \) defined in Remark 3, we have \( \tilde{Q}_{1/2, \gamma}(\beta) = \tilde{H}_\gamma(\beta) \).

The quantity \( \gamma \) is a shape parameter that controls the amount of robustness. The main reason for choosing a fixed (neither diminishing nor diverging) tuning parameter \( \gamma \) in Huber (1981) is to guarantee robustness towards arbitrary contamination in a neighborhood of the model. This is at the core of the robust statistics idiosyncrasy. In particular, Huber (1981) proposed \( \gamma = 1.35\sigma \) to gain as much robustness as possible while retaining 95\% asymptotic efficiency for normally distributed data, where \( \sigma > 0 \) is the standard deviation of the random noise. We estimate \( \sigma \) using the median absolute deviation on the residuals at each iteration, i.e., \( \text{MAD}(\{r_i^t\}_{i=1}^{n}) = \text{median}(|r_i^t - \text{median}(r_i^t)|) \).

Noting that the Huber loss is continuously differentiable, convex, and locally strongly convex, we use the GD-BB method described in the previous section to solve the optimization problem (27). Starting at iteration 0 with \( \tilde{\beta}^0 = 0 \), at iteration \( t = 0, 1, 2, \ldots \), we compute

\[ \tilde{\beta}^{t+1} = \tilde{\beta}^t - \eta_t \nabla \tilde{H}_\gamma(\tilde{\beta}^t) = \tilde{\beta}^t + \frac{\eta_t}{n} \sum_{i=1}^{n} \psi_\gamma(y_i - \langle x_i, \tilde{\beta}^t \rangle) x_i \] \hspace{1cm} (28)

with \( \eta_t > 0 \) automatically obtained by the BB method, where \( \psi_\gamma(u) = H_\gamma'(u) = \min\{\max(-\gamma, u), \gamma\} \).

We summarize the details in Algorithm 2

\begin{algorithm}
\begin{algorithmic}[1]
\Input \( \{(y_i, x_i)\}_{i=1}^{n} \) and convergence criterion \( \delta \).
\State Initialize \( \tilde{\beta}^{(0)} = 0 \)
\State Compute \( \gamma^0 = 1.35 \cdot \text{MAD}(\{r_i^0\}_{i=1}^{n}) \), where \( r_i^0 \leftarrow y_i - \langle x_i, \tilde{\beta}^0 \rangle \), \( i = 1, \ldots, n \), where \( \text{MAD}(\cdot) \) is the median absolute deviation
\State \( \tilde{\beta}^1 \leftarrow \tilde{\beta}^0 - \nabla \tilde{H}_\gamma(\tilde{\beta}^0) \)
\For {\( t = 1, 2, \ldots \)}
\State \( \gamma^t = 1.35 \cdot \text{MAD}(\{r_i^t\}_{i=1}^{n}) \), where \( r_i^t \leftarrow y_i - \langle x_i, \tilde{\beta}^t \rangle \), \( i = 1, \ldots, n \)
\State \( \delta^t \leftarrow \delta^t - \beta^t - \beta^{t-1}, g^t \leftarrow \nabla \tilde{H}_\gamma^t(\tilde{\beta}^t) - \nabla \tilde{H}_\gamma^{t-1}(\tilde{\beta}^{t-1}) \)
\State \( \eta_{1,t} \leftarrow \langle \delta^t, \delta^t \rangle / \langle \delta^t, g^t \rangle, \eta_{2,t} \leftarrow \langle \delta^t, g^t \rangle / \langle g^t, g^t \rangle \)
\State \( \eta_t \leftarrow \min\{\eta_{1,t}, \eta_{2,t}, 100\} \) if \( \eta_{1,t} > 0 \) and \( \eta_t \leftarrow 1 \) otherwise
\State \( \tilde{\beta}^{t+1} \leftarrow \tilde{\beta}^t - \eta_t \nabla \tilde{H}_\gamma^t(\tilde{\beta}^t) \)
\EndFor
\State \text{end for when } \|\nabla \tilde{H}_\gamma(\tilde{\beta})\|_2 \leq \delta
\end{algorithmic}
\end{algorithm}
via gradient descent with a shrinking gamma. To evaluate its performance, we implement
the above idea by setting $\gamma^t = c \cdot \gamma^{t-1}$ for some $c \in (0, 1)$ at the $t$-th iteration. We found that
the aforementioned idea is not stable numerically across several simulated data sets, unless
one is very careful in controlling the minimal magnitude of $\gamma$. In addition, the solution
obtained has a higher estimation error than that of standard QR and conquer.

5. Numerical Studies

In this section, we assess the finite-sample performance of conquer via extensive numerical
studies. We compare conquer to standard QR (Koenker and Bassett, 1978) and Horowitz’s
smoothed QR (Horowitz, 1998). Both the convolution-type and Horowitz’s smoothed meth-
ods involve a smoothing parameter $h$. In view of Theorem 3, we take $h = \{(p + \log n)/n\}^{2/5}$
in all of the numerical experiments. As we will see from Figure 5, the proposed method
is insensitive to the choice of $h$. Therefore, we leave the fine tuning of $h$ as an optional
rather than imperative choice. In all the numerical experiments, the convergence criterion
in Algorithms 1 and 2 is taken as $\delta = 10^{-4}$.

We first generate the covariates $x_i = (x_{i,1}, \ldots, x_{i,p})^T$ from a multivariate uniform dis-
tribution on the cube $3^{1/2} \cdot [-1, 1]^p$ with covariance matrix $\Sigma = (0.7^{|j-k|})_{1 \leq j,k \leq p}$ using the
R package MultRNG (Falk, 1999). The random noise $\varepsilon_i$ are generated from two different
distributions: (i) Gaussian distribution, $N(0, 4)$; and (ii) $t$ distribution with degrees of free-
dom 2, $t_2$. Let $x_i^- = (x_{i,1}, \ldots, x_{i,p-1})^T$, $\beta^* = (1, \ldots, 1)^T_{p-1}$, and $\beta^*_0 = 1$. Given $\tau \in (0, 1)$, we
then generate the response $y_i$ from the following homogeneous and heterogeneous models,
all of which satisfy the Assumption 1:

1. Homogeneous model:

$$y_i = \beta^*_0 + \langle x_i^-, \beta^* \rangle + \varepsilon_i - F_{\varepsilon_i}^{-1}(\tau), \quad i = 1, \ldots, n;$$

2. Linear heterogeneous model:

$$y_i = \beta^*_0 + \langle x_i^-, \beta^* \rangle + (0.5 x_{i,p} + 1) \{\varepsilon_i - F_{\varepsilon_i}^{-1}(\tau)\}, \quad i = 1, \ldots, n;$$

3. Quadratic heterogeneous model:

$$y_i = \beta^*_0 + \langle x_i^-, \beta^* \rangle + 0.5 \{1 + (x_{i,p} - 1)^2\} \{\varepsilon_i - F_{\varepsilon_i}^{-1}(\tau)\}, \quad i = 1, \ldots, n;$$

To evaluate the performance of different methods, we calculate the estimation error un-
der the $\ell_2$-norm, i.e., $||\hat{\beta} - \beta^*||_2$, and record the elapsed time. The details are in Section 5.1.
In Section 5.2, we examine the finite-sample performance of the multiplier bootstrap method
for constructing confidence intervals in terms of coverage probability, width of the interval,
and computing time.

5.1 Estimation

For all the numerical studies in this section, we consider a wide range of the sample size
$n$, with the size-dimension ratio fixed at $n/p = 20$. That is, we allow the dimension $p$
to increase as a function of $n$. We implement conquer with four different kernel functions as
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described in Remark 3: (i) Gaussian; (ii) uniform; (iii) Epanechnikov; and (iv) triangular. The classical quantile regression is implemented via a modified version of the Barrodale and Roberts algorithm (Koenker and d'Orey, 1987, 1994) by setting method = “br” in the R package quantreg, which is recommended for problems up to several thousands of observations in Koenker (2019). For very large problems, the Frisch-Newton approach after preprocessing “pfn” is preferred. Since the same size taken to be at most 5000 throughout this section, the two methods, “br” and “pfn”, have nearly identical runtime behaviors. In some applications where there are a lot of discrete covariates, it is advantageous to use method “sfn”, a sparse version of Frisch-Newton algorithm that exploits sparse algebra to compute iterates (Koenker and Ng, 2003). Moreover, we implement Horowitz's smoothed quantile regression using the Gaussian kernel, and solve the resulting non-convex optimization via gradient descent with random initialization and stepsize calibrated by backtracking line search (Section 9.3 of Boyd and Vandenberghe, 2004). The results, averaged over 100 replications, are reported.

Figure 3 depicts estimation error of the different methods under the simulation settings described in Section 5 with $\tau = 0.9$. We see that conquer has a lower estimation error than the classical QR across all scenarios, indicating that smoothing can improve estimation accuracy under the finite-sample setting. Moreover, compared to Horowitz's smoothing, conquer has a lower estimation error in most settings. Estimation error under various quantile levels $\tau \in \{0.1, 0.3, 0.5, 0.7\}$ under the $t_2$ random noise are also examined. The results are reported in Figure 9 in Appendix D, from which we observe evident advantages of conquer, especially at low and high quantile levels.

To assess the computational efficiency, we compute the elapsed time for fitting the different methods. Figures 4 and 10 in Appendix D report the runtime for the different methods with growing sample size and dimension under the same settings as in Figures 3 and 9, respectively. We observe that conquer is computationally efficient and stable across all scenarios, and the runtime is insensitive to the choice of kernel functions. In contrast, the runtime for classical quantile regression grows rapidly as the sample size and dimension increase. Figures 4 and 10 in Appendix D show that the runtime of Horowitz’s smoothing method increases significantly at extreme quantile levels $\tau \in \{0.1, 0.9\}$, possibly due to the combination of its non-convex nature and flatter gradient. In summary, we conclude that conquer significantly improves computational efficiency while retaining high statistical accuracy for fitting large-scale linear quantile regression models.

Next, we conduct a sensitivity analysis for conquer regarding the smoothing bandwidth $h$. We set $(n, p) = (2000, 100)$ and consider the simulation settings (29)–(31) with $N(0, 4)$ and $t_2$ noise. We perform conquer with $h \in \{0.2, 0.22, \ldots, 0.5\}$, including the default value $h_{de} = \{(p + \log n)/n\}^{2/5} = 0.3107$, and compare the estimation error with that of QR in Figure 5. We see that the estimation error of conquer is uniformly lower than that of QR over a range of $h$, suggesting that conquer is insensitive to the choice of bandwidth $h$.

5.2 Inference

In this section, we assess the performance of the multiplier bootstrap procedure for constructing confidence interval for each of the regression coefficients obtained from conquer. We implement conquer using the Gaussian kernel, and construct three types of confidence
Figure 3: Estimation error under models (29)–(31) in Section 5 with $\mathcal{N}(0, 4)$ and $t_2$ errors, $\tau = 0.9$, averaged over 100 data sets for three different methods: (i) quantile regression \texttt{qr}, (ii) Horowitz’ method with Gaussian kernel \texttt{Horowitz-Gauss}, and (iii) the conquer method with four different kernel functions \texttt{conquer-trian}, \texttt{conquer-para}, \texttt{conquer-unif}, and \texttt{conquer-Gauss}.

For the percentile \texttt{mb-per}; (ii) pivotal \texttt{mb-piv}; (iii) and regular \texttt{mb-norm} confidence intervals, as described in Section 2.4. We also refer to the proposed multiplier bootstrap procedure as \texttt{mb-conquer} for simplicity. We compare the proposed method to several widely used inference methods for QR. In particular, we consider confidence intervals by inverting a rank score test, \texttt{rank} (Gutenbrunner and Jurečková (1992); Section 3.5 of Koenker (2005)); a bootstrap variant based on pivotal estimating functions, \texttt{pwy} (Parzen, Wei and Ying, 1994); and wild bootstrap with Rademacher weights, \texttt{wild} (Feng, He and Hu, 2011). The three methods \texttt{rank}, \texttt{pwy}, and \texttt{wild} are implemented using the \texttt{R} package \texttt{quantreg}.

Note that \texttt{rank} is a non-resampling based procedure that relies on prior knowledge on the random noise, i.e., a user needs to specify whether the random noise are independent and identically distributed. In our simulation studies, we provide \texttt{rank} an unfair advantage by specifying the correct random noise structure.

We set $(n, p) = (800, 20)$, $\tau \in \{0.5, 0.9\}$, and significance level $\alpha = 0.05$. All of the resampling methods are implemented with $B = 1000$ bootstrap samples. To measure the reliability, accuracy, and computational efficiency of different methods for constructing confidence intervals, we calculate the average empirical coverage probability, average width of
The model settings are the same as those in Figure 3. Confidence interval, and the average runtime. The average is taken over all regression coefficients without the intercept. Results based on 200 replications are reported in Figure 6, and Figures 11–13 in Appendix D.

In Figure 6, and Figures 11–13 in Appendix D, we use the rank-inversion method, \texttt{rank}, as a benchmark since we implement \texttt{rank} using information about the true underlying random noise, which is practically infeasible. In the case of $\tau = 0.9$, \texttt{pwy} is most conservative as it produces the widest confidence intervals with slightly inflated coverage probability, and \texttt{wild} gives the narrowest confidence intervals but at the cost of coverage probability. The proposed methods \texttt{mb-per}, \texttt{mb-piv}, and \texttt{mb-norm} achieve a good balance between reliability (high coverage probability) and accuracy (narrow CI width), and moreover, has the lowest runtime.

To further highlight the computational gain of the proposed method, we now perform numerical studies with larger $n$ and $p$. In this case, the rank inversion method \texttt{rank} is computationally infeasible. For example, when $(n, p) = (5000, 250)$, rank inversion takes approximately 80 minutes while conquer with multiplier bootstrap takes 41 seconds for constructing confidence intervals. We therefore omit \texttt{rank} from the following comparison. We consider the quadratic heterogeneous model (31) with $(n, p) = (4000, 100)$ and $t_2$ noise. The results are reported in Figure 7. We see that \texttt{pwy} and \texttt{wild} take approximately 300
Figure 5: Sensitivity analysis of conquer with a range of bandwidth parameter $h$. Results for $(n,p) = (2000, 100)$, averaged over 100 datasets, with conquer implemented using a Gaussian kernel. The blue vertical dash line represents our default choice $h_{de} = \{(p+\log n)/n\}^{2/5}$, and the red horizontal dash line represents the estimation error of standard QR.

seconds while \texttt{mb-conquer} takes approximately 15 seconds. In summary, \texttt{mb-conquer} leads to a huge computational gain without sacrificing statistical efficiency.

### 5.3 Comparison between normal approximation and bootstrap calibration

Finally, we complement the above studies with a comparison between the normal approximation and bootstrap calibration methods for confidence estimation. We consider model (29) with $(n, p) = (2000, 10)$. We use the same $\beta^* \in \mathbb{R}^{p-1}$ and $\beta_0^*$ as before, and generate random covariates and noise from a multivariate uniform distribution and $t_{1.5}$-distribution, respectively. For each of the $p-1$ regression coefficients, we apply the proposed bootstrap percentile method and the normal-based method (Fernandes, Guerre and Horta, 2019) to construct pointwise confidence intervals at quantile indices close to 0 and 1, that is, $\tau \in \{0.05, 0.1, 0.9, 0.95\}$.

Boxplots of the empirical coverage and CI width for the two methods are reported in Figure 8. Considering that extreme quantile regressions are notoriously hard to estimate, the bootstrap method can produce much more reliable (high coverage) and accurate (narrow
Figure 6: Empirical coverage, confidence interval width, and elapsed time of six methods: rank, pwy, wild and three types of mb-conquer: mb-per, mb-piv, and mb-norm under models (29)–(31) with $t_2$ errors. For the running time, rank is not included since it is not a resampling-based method. The quantile level $\tau$ is fixed to be 0.9, and the results are averaged over 200 data sets.

width) confidence intervals than the normal-based counterpart. Therefore, for applications in which extreme quantiles are of particular interest, such as the problem of forecasting the conditional value-at-risk of a financial institution (Chernozhukov and Umantsev, 2001), the bootstrap provides a more reliable approach for quantifying the uncertainty of the estimates.
For more information, please refer to the full text.
In the high-dimensional setting in which $p \gg n$, various authors have studied the regularized quantile regression under the sparsity assumption that most of the regression coefficients are zero (Belloni and Chernozhukov, 2011; Wang, Wu and Li, 2012). Using $\ell_1$ penalties, the computation of regularized QR is based on either a linear programming reformulation or alternating direction method of multiplier algorithms (Gu et al., 2018). Since the conquer loss is convex and twice differentiable, we expect that gradient-based algorithms, such as coordinate gradient descent or proximal gradient descent, will enjoy superior computational efficiency for solving regularized conquer without sacrificing statistical accuracy. One future work is to establish the statistical theory and computational complexity of the regularized conquer estimator.

References


**Appendix A. One-step Conquer with Higher-order Kernels**

As noted in Section 3.1, the smoothing bias is of order $h^2$ when a non-negative kernel is used. The ensuing empirical loss $\beta \mapsto (1/n) \sum_{i=1}^{n} (\rho_{\tau} * K_h)(y_i - \langle x_i, \beta \rangle)$ is not only twice-differentiable and convex, but also (provably) strongly convex in a local vicinity of $\beta^*$ with high probability. Kernel smoothing is ubiquitous in nonparametric statistics. The order of a kernel, $\nu$, is defined as the order of the first non-zero moment. The order of a symmetric kernel is always even. A kernel is called *high-order* if $\nu > 2$, which inevitably has negative parts and thus is no longer a probability density. Thus far we have focused on conquer with second-order kernels, and the resulting estimator achieves an $\ell_2$-error of the order $\sqrt{p/n} + h^2$.

Let $G(\cdot)$ be a higher-order symmetric kernel with order $\nu \geq 4$, and $b > 0$ be a bandwidth. Again, via convolution smoothing, we may consider a bias-reduced estimator that minimizes the empirical loss $\beta \mapsto \hat{Q}_G^b(\beta) := (1/n) \sum_{i=1}^{n} (\rho_{\tau} * G_b)(y_i - \langle x_i, \beta \rangle)$. This, however, leads to a non-convex optimization. Without further assumptions, finding a global minimum is computationally intractable: finding an $\epsilon$-suboptimal point for a $k$-times continuously differentiable loss function requires at least $\Omega\{(1/\epsilon)^{p/k}\}$ evaluations of the function and its
first $k$ derivatives, ignoring problem-dependent constants; see Section 1.6 in Nemirovskii and Yudin (1983). Instead, various gradient-based methods have been developed for computing stationary points, which are points $\beta$ with sufficiently small gradient $\|\nabla \tilde{Q}_b^G(\beta)\|_2 \leq \epsilon$, where $\epsilon \geq 0$ is optimization error. However, the equation $\nabla \tilde{Q}_b^G(\beta) = 0$ does not necessarily have a unique solution, whose statistical guarantees remain unknown.

Motivated by the classical one-step estimator (Bickel, 1975), we further propose a one-step conquer estimator using high-order kernels, which bypasses solving a large-scale non-convex optimization. To begin with, we choose two symmetric kernel functions, $K : \mathbb{R} \to [0, \infty)$ with order two and $G(\cdot)$ with order $\nu \geq 4$, and let $h, b > 0$ be two bandwidths. First, compute an initial conquer estimator $\tilde{\beta} \in \text{argmin}_{\beta \in \mathbb{R}^p} \tilde{Q}_h^K(\beta)$, where $\tilde{Q}_h^K(\beta) = (1/n) \sum_{i=1}^n (\rho_l \ast K_h)(y_i - \langle x_i, \beta \rangle)$. Denote by $\hat{r}_i = y_i - \langle x_i, \bar{\beta} \rangle$ for $i = 1, \ldots, n$ the fitted residuals. Next, with slight abuse of notation, we define the one-step conquer estimator $\hat{\beta}$ as a solution to the equation $\nabla^2 \tilde{Q}_b^G(\tilde{\beta})(\hat{\beta} - \bar{\beta}) = -\nabla \tilde{Q}_b^G(\bar{\beta})$, or equivalently,

$$\left\{ \frac{1}{n} \sum_{i=1}^n G_b(\hat{r}_i) x_i x_i^T \right\}(\hat{\beta} - \bar{\beta}) = \frac{1}{n} \sum_{i=1}^n \{ \tilde{G}(\hat{r}_i/b) + \tau - 1 \} x_i.$$  \hspace{1cm} (32)

where $\tilde{Q}_b^G(\beta) = (1/n) \sum_{i=1}^n (\rho_l \ast G_b)(y_i - \langle x_i, \beta \rangle)$. Provided that $\nabla^2 \tilde{Q}_b^G(\bar{\beta})$ is positive definite, the one-step conquer estimate $\hat{\beta}$ essentially performs a Newton-type step based on $\bar{\beta}$:

$$\hat{\beta} = \bar{\beta} - \{ \nabla^2 \tilde{Q}_b^G(\bar{\beta}) \}^{-1} \nabla \tilde{Q}_b^G(\bar{\beta}).$$  \hspace{1cm} (33)

In this case, $\hat{\beta}$ can be computed by the conjugate gradient method (Hestenes and Stiefel, 1952).

Theoretical properties of the one-step estimator $\hat{\beta}$ defined in (32), including the Bahadur representation and asymptotic normality with explicit Berry-Esseen bound, will be provided in on-line supplementary materials. For practical implementation, we consider higher-order Gaussian-based kernels. For $r = 1, 2, \ldots$, the $(2r)$-th order Gaussian kernels are

$$G_{2r}(u) = \frac{(-1)^r \phi^{(2r-1)}(u)}{2^{2r-1} (r-1)! u} = \sum_{\ell=0}^{r-1} \frac{(-1)^\ell}{2^{\ell} \ell!} \phi^{(2\ell)}(u);$$

see Section 2 of Wand and Schucany (1990). Integrating $G_{2r}(\cdot)$ yields

$$\tilde{G}_{2r}(v) = \int_{-\infty}^v G_{2r}(u) \, du = \sum_{\ell=0}^{r-1} \frac{(-1)^\ell}{2^{\ell} \ell!} \phi^{(2\ell-1)}(v).$$

In fact, both $G_{2r}$ and $\tilde{G}_{2r}$ have simpler forms $G_{2r}(u) = p_r(u) \phi(u)$ and $\tilde{G}_{2r}(u) = \Phi(u) + P_r(u) \phi(u)$, where $p_r(\cdot)$ and $P_r(\cdot)$ are polynomials in $u$. For example, $p_1(u) = 1$, $P_1(u) = 0$, $p_2(u) = (-u^2 + 3)/2$, $P_2(u) = u/2$, $p_3(u) = (u^4 - 10u^2 + 15)/8$, and $P_3(u) = (-u^3 + 7u)/8$. We refer to Oryshchenko (2020) for more details when $r$ is large.

**Appendix B. Proofs**

Recall that $\Sigma = \mathbb{E}(xx^T)$ is positive definite. Throughout the proof, we write $\Omega = \Sigma^{-1}$ and $\|u\|_\Omega = \|\Sigma^{-1/2}u\|_2$ for $u \in \mathbb{R}^p$. By Hölder’s inequality, $|\langle u, v \rangle| \leq \|u\|_{\Sigma} \cdot \|v\|_{\Omega}$ for any $u, v \in \mathbb{R}^p$.  

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B.1 Proof of Proposition 1

For every \( r > 0 \), define the ellipse \( \Theta(r) = \{ u \in \mathbb{R}^p : \| u \| \Sigma \leq r \} \) and the local vicinity \( \Theta^* = \{ \beta \in \mathbb{R}^p : \beta - \beta^* \in \Theta(\kappa_2^{1/2}) \} \). Let \( \eta = \sup \{ u \in [0, 1] : u(\beta^*_e - \beta^*) \in \Theta(\kappa_2^{1/2}) \} \) and \( \tilde{\beta}^* = \beta^* + \eta(\beta^*_e - \beta^*) \). By definition, \( \eta = 1 \) if \( \beta^*_e \in \Theta^* \) and \( \eta < 1 \) if \( \beta^*_e \notin \Theta^* \). In the latter case, \( \beta^* \in \partial \Theta^* \). By the convexity of \( \beta \mapsto Q_h(\beta) \) and Lemma C.1 in the supplementary material of Sun, Zhou and Fan (2020),

\[
0 \leq \langle \nabla Q_h(\beta^*) - \nabla Q_h(\beta^*), \beta^* - \beta^* \rangle \\
\leq \eta \cdot \langle \nabla Q_h(\beta^*_e) - \nabla Q_h(\beta^*), \beta^*_e - \beta^* \rangle = \langle -\nabla Q_h(\beta^*), \beta^* - \beta^* \rangle. \tag{34}
\]

It follows from the mean value theorem for vector-valued functions that

\[
\nabla Q_h(\beta^*) - \nabla Q_h(\beta^*) = \int_0^1 \nabla^2 Q_h((1-t)\beta^* + t\beta^*) \, dt \tag{35}
\]

where \( \nabla^2 Q_h(\beta) = \mathbb{E}\{ K_h(y - (x, \beta)) (x \Sigma^T) \} \) for \( \beta \in \mathbb{R}^p \). With \( \delta = \beta - \beta^* \), note that

\[
\mathbb{E}\{ K_h(y - (x, \beta)) | x \} = \frac{1}{h} \int_{-\infty}^{\infty} K\left( \frac{u - (x, \delta)}{h} \right) f_\varepsilon(x) \, du = \int_{-\infty}^{\infty} K(v) f_\varepsilon(x) \langle \langle x, \delta \rangle + hv \rangle \, dv.
\]

By the Lipschitz continuity of \( f_\varepsilon(x) \),

\[
\mathbb{E}\{ K_h(y - (x, \beta)) | x \} = f_\varepsilon(x)(0) + R_h(\delta) \tag{36}
\]

with \( R_h(\delta) \) satisfying \( |R_h(\delta)| \leq l_0 |\langle x, \delta \rangle| + \kappa_1 h \). Substituting (36) into (34) and (35) yields

\[
\langle \nabla Q_h(\beta^*) - \nabla Q_h(\beta^*), \beta^* - \beta^* \rangle \\
\geq \| \beta^* - \beta^* \|_D^2 - 0.5l_0 \mathbb{E}|\langle x, \beta^* - \beta^* \rangle|^3 - l_0 \kappa_1 h \cdot \| \beta^* - \beta^* \|_\Sigma^2 \\
\geq \| \beta^* - \beta^* \|_D^2 - 0.5l_0 m_3 \cdot \| \beta^* - \beta^* \|_\Sigma^2 - l_0 \kappa_1 h \cdot \| \beta^* - \beta^* \|_\Sigma^2. \tag{37}
\]

On the other hand, under model (11) we have

\[
\langle -\nabla Q_h(\beta^*), \beta^* - \beta^* \rangle \leq \| \nabla Q_h(\beta^*) \|_\Omega \cdot \| \beta^* - \beta^* \|_\Sigma,
\]

where \( \nabla Q_h(\beta^*) = \mathbb{E}\{ K(-\varepsilon/h) - \tau \} x \). By integration by parts and a Taylor series expansion,

\[
\mathbb{E}\{ K(-\varepsilon/h) | x \} = \int_{-\infty}^{\infty} K(-t/h) \, dF_\varepsilon(t) \\
= -\frac{1}{h} \int_{-\infty}^{\infty} K(-t/h) F_\varepsilon(t) \, dt = \int_{-\infty}^{\infty} K(u) F_\varepsilon(-hu) \, du \\
= \tau + \int_{-\infty}^{\infty} K(u) \int_{-hu}^{0} \{ f_\varepsilon(t) - f_\varepsilon(0) \} \, dt \, du,
\]

from which it follows that \( |\mathbb{E}\{ K(-\varepsilon/h) | x \} - \tau| \leq 0.5l_0 \kappa_2 h^2 \). Consequently,

\[
\| \nabla Q_h(\beta^*) \|_\Omega = \sup_{u \in \mathbb{R}^{p-1}} \mathbb{E}\{ K(-\varepsilon/h) - \tau \} \Sigma^{-1/2} x, u \leq 0.5l_0 \kappa_2 h^2. \tag{38}
\]
Putting together the pieces, we conclude that
\[
\langle -\nabla Q_h(\beta^*), \bar{\beta}^* - \beta^* \rangle \leq 0.5 \lambda_0 \kappa_2 h^2 \cdot \|\bar{\beta}^* - \beta^*\|_\Sigma. \tag{39}
\]

Recall that \( f_{\epsilon|x}(0) \geq f > 0 \) almost surely and \( \|\bar{\beta}^* - \beta^*\|_\Sigma \leq \kappa_2 h^2 \). Combining this with (34), (37) and (39) yields
\[
f \cdot \|\bar{\beta}^* - \beta^*\|_\Sigma \leq \|\bar{\beta}^* - \beta^*\|_D \leq (0.5m_3 \kappa_2 + 0.5 \kappa_2 + \kappa_1 \kappa_2^{1/2}) \lambda_0 h^2 \cdot \|\bar{\beta}^* - \beta^*\|_\Sigma.
\]

Canceling \( \|\bar{\beta}^* - \beta^*\|_\Sigma \) on both sides, we obtain
\[
\|\bar{\beta}^* - \beta^*\|_\Sigma \leq \frac{f}{\lambda_0 \kappa_1 h^2} \left( 0.5m_3 \kappa_2^{1/2} + 0.5 \kappa_2^{1/2} + \kappa_1 \right) \lambda_0 h \cdot \kappa_2^{1/2} h = \frac{1}{\kappa_1 h} c_K \lambda_0 h \cdot \kappa_2^{1/2} h.
\]

Provided that \( h < f/(c_K \lambda_0) \), \( \bar{\beta}^* \) falls into the interior of \( \Theta^* \), i.e., \( \|\bar{\beta}^* - \beta^*\|_\Sigma < \kappa_2^{1/2} h \), thus enforcing \( \eta = 1 \) (otherwise, by construction \( \bar{\beta}^* \) must lie on the boundary which leads to contradiction) and hence \( \beta^* = \beta_h^* \). In addition, by (37), \( Q_h(\cdot) \) is strictly convex in a neighborhood of \( \beta_h^* \) so that \( \beta_h^* \) is the unique minimizer and satisfies the stated bound (13).

Next, to investigate the leading term in the bias, define
\[
\Delta = \Sigma^{-1/2} \{ \nabla Q_h(\beta^*_h) - \nabla Q_h(\beta^*) - D(\beta^*_h - \beta^*) \} \quad \text{and} \quad D_0 = \Sigma^{-1/2} D \Sigma^{-1/2} = \mathbb{E} \{ f_{\epsilon|x}(0) w w^T \},
\]
where \( w = \Sigma^{-1/2} x \). Again, by the mean value theorem for vector-valued functions,
\[
\Delta = \left\{ \Sigma^{-1/2} \int_0^1 \nabla^2 Q_h((1-t)\beta^* + t\beta^*_h) \, dt \cdot \Sigma^{-1/2} - D_0 \right\} \Sigma^{1/2}(\beta^*_h - \beta^*). \tag{40}
\]

The Lipschitz continuity of \( f_{\epsilon|x}(\cdot) \) ensures that
\[
\left\| \Sigma^{-1/2} \int_0^1 \nabla^2 Q_h((1-t)\beta^* + t\beta^*_h) \, dt \cdot \Sigma^{-1/2} - D_0 \right\|_2 \leq \lambda_0 \sup_{u \in \mathbb{S}^{p-1}} \mathbb{E} \int_0^1 \int_{-\infty}^\infty K(u) \{ f_{\epsilon|x}(t\langle x, \beta^*_h - \beta^* \rangle + hu) - f_{\epsilon|x}(0) \} \, du \, dt \, w w^T \|_2
\leq \lambda_0 \sup_{u \in \mathbb{S}^{p-1}} \mathbb{E} \int_0^1 \int_{-\infty}^\infty K(u) \{ |t\langle x, \beta^*_h - \beta^* \rangle + hu| |u| \} \, du \, dt \langle w, u \rangle^2
\leq 0.5 \lambda_0 \sup_{u \in \mathbb{S}^{p-1}} \mathbb{E} \{ |\langle x, \beta^*_h - \beta^* \rangle| |w, u|^2 \} + \lambda_0 \kappa_1 h
\leq 0.5 \lambda_0 m_3 \|\beta^*_h - \beta^*\|_\Sigma + \lambda_0 \kappa_1 h.
\]

This bound, together with (40), implies
\[
\|\Delta\|_2 \leq \lambda_0 (0.5m_3 \|\beta^*_h - \beta^*\|_\Sigma + \kappa_1 h) \|\beta^*_h - \beta^*\|_\Sigma. \tag{41}
\]
Moreover, applying a second-order Taylor series expansion to \( f_{\varepsilon|x}(\cdot) \) yields

\[
E\{R(-\varepsilon/h|x)\} - \tau
= \int_{-\infty}^{\infty} K(u) \int_0^{hu} \{f_{\varepsilon|x}(t) - f_{\varepsilon|x}(0)\} \, dt \, du
= 0.5\kappa_2 h^2 \cdot f'_{\varepsilon|x}(0) + \int_{-\infty}^{\infty} \int_0^{hu} K(u) \{f'_{\varepsilon|x}(v) - f'_{\varepsilon|x}(0)\} \, dv \, dt \, du.
\]

For \( \nabla Q_h(\beta^*) = E\{R(-\varepsilon/h) - \tau\}x \), it follows that

\[
\left\| \nabla Q_h(\beta^*) - \frac{1}{2} \kappa_2 h^2 \cdot E\{f'_{\varepsilon|x}(0)x\} \right\|_{\Omega} \leq \frac{1}{6} l_1 \kappa_3 h^3.
\]  \( \square \)

\[ \text{B.2 Proof of Theorem 1} \]

Recall from (6) that \( \hat{\beta}_h \) is the smoothed quantile regression estimator obtained by minimizing \( \tilde{Q}_h(\cdot) \). For any given \( \tau \in (0, 1) \) and \( h > 0 \), by the optimality of \( \hat{\beta}_h \) and convexity of \( \tilde{Q}_h(\cdot) \), we have \( \nabla \tilde{Q}_h(\hat{\beta}_h) = 0 \) and \( \langle \nabla \tilde{Q}_h(\hat{\beta}_h) - \nabla \tilde{Q}_h(\beta^*) , \hat{\beta}_h - \beta^* \rangle \geq 0 \).

As in the proof of Proposition 1, we write \( \Theta(t) = \{u \in \mathbb{R}^p : \|u\|_\Sigma \leq t\} \) for \( t \geq 0 \).

For some \( r > 0 \) to be determined, define \( \bar{\beta} = \beta^* + \eta(\hat{\beta}_h - \beta^*) \) where \( \eta := \sup\{u \in [0, 1] : u(\hat{\beta}_h - \beta^*) \in \Theta(r)\} \). Thus, by definition, \( \eta = 1 \) if \( \hat{\beta}_h \in \beta^* + \Theta(r) \), and \( \eta < 1 \) if \( \hat{\beta}_h \notin \beta^* + \Theta(r) \). In the latter case, we must have \( \beta \in \beta^* + \partial \Theta(r) \), where \( \partial \Theta(r) \) denotes the boundary of \( \Theta(r) \). Consider the symmetrized Bregman divergence associated with \( \hat{Q}_h(\cdot) \):

\[
D(\beta_1, \beta_2) = \langle \nabla \hat{Q}_h(\beta_1) - \nabla \hat{Q}_h(\beta_2), \beta_1 - \beta_2 \rangle, \quad \beta_1, \beta_2 \in \mathbb{R}^p.
\]  \( \text{(43)} \)

By Lemma C.1 in the supplement of Sun, Zhou and Fan (2020), the three points \( \hat{\beta}_h, \bar{\beta}, \) and \( \beta^* \) satisfy \( D(\hat{\beta}_h, \beta^*) \leq \eta D(\bar{\beta}, \beta^*) \). Combined with the facts \( \bar{\beta} - \beta^* = \eta(\hat{\beta}_h - \beta^*) \in \Theta(r) \) and \( \nabla \tilde{Q}_h(\hat{\beta}_h) = 0 \), we find that

\[
\|\beta - \beta^*\|_\Sigma^2 \cdot \frac{D(\hat{\beta}_h, \beta^*)}{\|\beta - \beta^*\|_\Sigma^2} \leq -\eta \langle \nabla \tilde{Q}_h(\beta^*), \hat{\beta}_h - \beta^* \rangle \leq \|\nabla \hat{Q}_h(\beta^*)\|_{\Omega} \cdot \|\beta - \beta^*\|_\Sigma.
\]

Canceling \( \|\beta - \beta^*\|_\Sigma \) on both sides yields

\[
\|\beta - \beta^*\|_\Sigma \leq \frac{\|\nabla \hat{Q}_h(\beta^*)\|_{\Omega}}{\inf_{\beta \in \beta^* + \Theta(r)} \{D(\beta, \beta^*) / \|\beta - \beta^*\|_\Sigma^2\}}.
\]  \( \text{(44)} \)

The following two lemmas provide, respectively, upper and lower bounds on \( \|\nabla \hat{Q}_h(\beta^*)\|_{\Omega} \) and \( \inf_{\beta \in \beta^* + \Theta(r)} \{D(\beta, \beta^*) / \|\beta - \beta^*\|_\Sigma^2\} \).

**Lemma 1.** Assume that Conditions 3.1-3.4 hold. For any \( t \geq 0 \),

\[
\|\nabla \hat{Q}_h(\beta^*)\|_{\Omega} \leq 1.46u_0 \left\{ v_1 \sqrt{\frac{4p + 2t}{n} + 2 \max(1 - \tau, \tau) \frac{2p + t}{n}} \right\} + 0.5l_0 \kappa_2 h^2
\]  \( \text{(45)} \)

with probability at least \( 1 - e^{-t} \), where \( v_1^2 = \tau(1 - \tau) + (1 + \tau)l_0 \kappa_2 h^2 \).
Under Condition 3.2, there exist constants $\tilde{f}_h \geq f_h$ such that

$$f_h \leq \inf_{|u| \leq h/2} f_{\xi|x}(u) \leq \sup_{|u| \leq h/2} f_{\xi|x}(u) \leq \tilde{f}_h$$  \hspace{1cm} (46)

almost surely (over $x$). By the Lipschitz continuity, we can take $\tilde{f}_h = \tilde{f} + l_0 h/2$ and $f_h = f - l_0 h/2$. Throughout the following, we assume (46) holds. For every $\delta \in (0, 1]$, we define $\eta_\delta \geq 0$ as

$$\eta_\delta = \inf \{ \eta > 0 : \mathbb{E}\{ (u, w)^2 : |(u, w)| > \eta_\delta \} \leq \delta \} \quad \forall u \in \mathbb{S}^{p-1},$$  \hspace{1cm} (47)

where $w = \Sigma^{-1/2} x$ is the standardized covariate vector that satisfies $\mathbb{E}(ww^T) = I_p$, and hence $\mathbb{E}(uu^T) = 1$ for any $u \in \mathbb{S}^{p-1}$. It can be shown that $\eta_\delta$ depends only on $\delta$ and $v_0$ in Condition 3.4, and the map $\delta \mapsto \eta_\delta$ is non-increasing with $\eta_\delta \downarrow 0$ as $\delta \uparrow 1$.

**Lemma 2.** Let $0 < h < 2/lf/l_0$ and $0 < r \leq h/(4\eta_1 h)$ with $\eta_1 h$ defined in (47). Then, for any $t > 0$,

$$\inf_{\beta \in \beta^* + \Theta(r)} \frac{D(\beta, \beta^*)}{\kappa\|\beta - \beta^*\|_\Sigma} \geq \frac{3}{4} f_h - \tilde{f}_h^{1/2} \left( \frac{5}{4} \sqrt{\frac{hp}{r^2 n}} + \sqrt{\frac{ht}{8r^2 n}} \right) - \frac{ht}{3r^2 n}$$  \hspace{1cm} (48)

with probability at least $1 - e^{-t}$.

In view of (44), (45), and (48), we take $r = \eta_1 h/(4\eta_1 h)$ so that as long as $(p + t)/n \leq h \leq 1$,

$$\|\tilde{\beta} - \beta^*\|_\Sigma \leq 3(\kappa_l)^{-1} v_0 \left\{ C_1^{1/2} \sqrt{\frac{4p + 2t}{n}} + 2 \max(1 - \tau, \tau) \frac{2p + t}{n} \right\} + (\kappa_l)^{-1} l_0 \kappa_2 h^2$$  \hspace{1cm} (49)

with probability at least $1 - 2e^{-t}$. With this choice of $r$, we see that under the constraint $\sqrt{(p + t)/n} \leq h \leq 1$, the right-hand side of (49) is strictly less than $r$. In other words, conditioned on an event that occurs with high probability, $\tilde{\beta}$ falls into the interior of $\beta^* + \Theta(r)$, thus enforcing $\eta = 1$ and $\tilde{\beta}_h = \beta$. The claimed bound for $\tilde{\beta}_h$ then follows immediately.

**B.2.1 Proof of Lemma 1**

Write $Q_h(\beta) = \mathbb{E}\{ \tilde{Q}_h(\beta) \}$, and define $\xi_i = \mathcal{K}(-\varepsilon_i/h) - \tau$ for $i = 1, \ldots, n$. By the triangle inequality and (38), we have

$$\|\nabla \tilde{Q}_h(\beta^*)\|_{\Omega} \leq \|\nabla \tilde{Q}_h(\beta^*) - \nabla Q_h(\beta^*)\|_{\Omega} + \|\nabla Q_h(\beta^*)\|_{\Omega} \leq \|\Sigma^{-1/2}\{\nabla \tilde{Q}_h(\beta^*) - \nabla Q_h(\beta^*)\}\|_2 + 0.5l_0 \kappa_2 h^2.$$  \hspace{1cm}

It suffices to obtain an upper bound for the centered score $\Sigma^{-1/2}\{\nabla \tilde{Q}_h(\beta^*) - \nabla Q_h(\beta^*)\} = (1/n) \sum_{i=1}^n \{\xi_i w_i - \mathbb{E}(\xi_i w_i)\} \in \mathbb{R}^p$, where $w_i = \Sigma^{-1/2} x_i$.

Using a covering argument, for any $\varepsilon \in (0, 1)$, there exists an $\varepsilon$-net $N_\varepsilon$ of the unit sphere with cardinality $|N_\varepsilon| \leq (1 + 2/\varepsilon)^p$ such that

$$\|\Sigma^{-1/2}\{\nabla \tilde{Q}_h(\beta^*) - \nabla Q_h(\beta^*)\}\|_2 \leq (1 - \varepsilon)^{-1} \max_{u \in N_\varepsilon} \langle u, \Sigma^{-1/2}\{\nabla \tilde{Q}_h(\beta^*) - \nabla Q_h(\beta^*)\} \rangle.$$
For each unit vector \( \mathbf{u} \in \mathcal{N}_\varepsilon \), define centered random variables \( \gamma_{u,i} = \langle \mathbf{u}, \xi_i \mathbf{w}_i - \mathbb{E}(\xi_i \mathbf{w}_i) \rangle \). It is easy to see that \( |\xi_i| \leq \tilde{\tau} := \max(1 - \tau, \tau) \). To bound \( \mathbb{E}(\xi_i^2 | x_i) \), by a change of variable and integration by parts,

\[
\mathbb{E}\{|\mathbf{K}(\varepsilon/h)|x\} = 2 \int_{-\infty}^{\infty} K(v)K(v)F_{\varepsilon|x}(-vh)dv
\]

\[
= 2\tilde{\tau} \int_{-\infty}^{\infty} K(v)K(v)dv - 2h\varepsilon f_{\varepsilon|x}(0) \int_{-\infty}^{\infty} vK(v)K(v)dv
\]

\[
+ 2 \int_{-\infty}^{\infty} \int_0^{-vh} \{f_{\varepsilon|x}(t) - f_{\varepsilon|x}(0)\}K(v)K(v)dvdt
\]

\[
\leq \tilde{\tau} + l_0 \kappa_2 h^2,
\]

where \( \kappa_2 \) and \( l_0 \) are the constants from Conditions 3.1 and 3.2. It then follows that \( \mathbb{E}(\xi_i^2 | x_i) \leq \tilde{\tau}^2 := (1 - \tau) + (1 + \tau)l_0 \kappa_2 h^2 \). Hence, for \( k = 2, 3, \ldots \),

\[
\mathbb{E}(\mathbb{E}(\mathbf{u}, \xi_i \mathbf{w}_i)^k) \leq \tilde{\tau}^{k-2} \mathbb{E}\{\mathbb{E}(\mathbf{u}, \xi_i \mathbf{w}_i)^{k} \cdot \mathbb{E}(\xi_i^2 | x_i)\}
\]

\[
\leq \tilde{\tau}^{k-2} \mathbb{E}\{\mathbb{E}(\mathbf{u}, \xi_i \mathbf{w}_i)^{k} \cdot \mathbb{E}(\xi_i^2 | x_i)\}
\]

\[
\leq v_0^2 \mathbb{E}(\mathbf{u}, \xi_i \mathbf{w}_i)^{k} \int_0^{\infty} P(|\mathbf{u}, \xi_i \mathbf{w}_i| \geq v_0 t)kt^{k-1}dt
\]

\[
\leq v_0^2 \mathbb{E}(\mathbf{u}, \xi_i \mathbf{w}_i)^{k} \int_0^{\infty} \frac{kt^{k-1}e^{-t}}{t}dt
\]

\[
= k! \cdot v_0^2 \mathbb{E}(\mathbf{u}, \xi_i \mathbf{w}_i)^{k-2}
\]

\[
\leq \frac{k!}{2} \cdot (v_\tau v_0)^2 \cdot (2\tilde{\tau} v_0)^{k-2}.
\]

Consequently, it follows from Bernstein’s inequality that for every \( u \geq 0 \),

\[
\frac{1}{n} \sum_{i=1}^{n} \gamma_{u,i} \leq v_0 \left(v_\tau \sqrt{\frac{2u}{n}} + \frac{2\tilde{\tau} u}{n}\right)
\]

with probability at least \( 1 - e^{-u} \).

Finally, applying a union bound over \( \mathbf{u} \in \mathcal{N}_\varepsilon \) yields

\[
\|\nabla \tilde{Q}_h(\mathbf{\beta}^*) - \nabla Q_h(\mathbf{\beta}^*)\|_\Omega \leq \frac{v_0}{1 - \epsilon} \left(v_\tau \sqrt{\frac{2u}{n}} + \frac{2\tilde{\tau} u}{n}\right)
\]

with probability at least \( 1 - e^{\log(1 + 2/\epsilon)p - u} \). Taking \( \epsilon = 2/(e^2 - 1) \) and \( u = 2p + t \) \( (t \geq 0) \) proves the claimed result. \( \square \)

**B.2.2 Proof of Lemma 2**

Recall that the empirical loss \( \tilde{Q}_h(\cdot) \) in (5) is convex and twice continuously differentiable with \( \nabla \tilde{Q}_h(\mathbf{\beta}) = (1/n) \sum_{i=1}^{n} [R((\mathbf{x}_i, \mathbf{\beta}) - y_i)/h] - \tau] \mathbf{x}_i \) and \( \nabla^2 \tilde{Q}_h(\mathbf{\beta}) = (1/n) \sum_{i=1}^{n} K_h((\mathbf{x}_i, \mathbf{\beta}) - y_i) \mathbf{x}_i \mathbf{x}_i^T \). For the symmetrized Bregman divergence \( D : \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^+ \) defined in (43), we have

\[
D(\mathbf{\beta}, \mathbf{\beta}^*) = \frac{1}{n} \sum_{i=1}^{n} \left\{ R \left( \frac{\langle \mathbf{x}_i, \mathbf{\beta} \rangle - y_i}{h} \right) - R \left( \frac{-\varepsilon_i}{h} \right) \right\} \langle \mathbf{x}_i, \mathbf{\beta} - \mathbf{\beta}^* \rangle.
\]  

(50)
Define the events $E_i = \{|\varepsilon_i| \leq h/2\} \cap \{|(x_i, \beta - \beta^*)| \leq \|\beta - \beta^*\|_2 \cdot h/(2r)\}$ for $i = 1, \ldots, n$. For any $\beta \in \beta^* + \Theta(r)$, note that $|y_i - (x_i, \beta)| \leq h$ on $E_i$, implying

$$D(\beta, \beta^*) \geq \frac{\kappa_i}{nh} \sum_{i=1}^{n} (x_i, \beta - \beta^*)(x_i, \beta - \beta^*) I_{E_i}, \quad (51)$$

where $I_{E_i}$ is the indicator function of $E_i$ and $\kappa_i = \min_{|u| \leq 1} K(u)$. It then suffices to bound the right-hand side of the above inequality from below uniformly over $\beta \in \beta^* + \Theta(r)$.

For $R > 0$, define the function $\varphi_R(u) = u^2 I(|u| \leq R/2) + \{u \text{ sign}(u) - R\}^2 I(R/2 < |u| \leq R)$, which is $R$-Lipschitz continuous and satisfies

$$u^2 I(|u| \leq R/2) \leq \varphi_R(u) \leq u^2 I(|u| \leq R). \quad (52)$$

Moreover, note that $\varphi_c(cu) = c^2 \varphi_R(u)$ for any $c > 0$ and $\varphi_0(u) = 0$. For $\beta \in \beta^* + \Theta(r)$, consider a change of variable $\delta = \Sigma^{1/2}(\beta - \beta^*)/\|\beta - \beta^*\|_2 \in \mathbb{S}^{p-1}$. Together, (51) and (52) imply

$$\frac{D(\beta, \beta^*)}{\|\beta - \beta^*\|_2^2} \geq \kappa_i \cdot \frac{1}{nh} \sum_{i=1}^{n} \omega_i \cdot \varphi_{1/(2r)}((w_i, \delta)), \quad (53)$$

where $\omega_i := I(|\varepsilon_i| \leq h/2)$ with $\varepsilon_i = y_i - (x_i, \beta^*)$, and $w_i = \Sigma^{-1/2} x_i$.

Next, we bound the expectation $E\{D_0(\delta)\}$ and the random fluctuation $D_0(\delta) - E\{D_0(\delta)\}$, separately, starting with the former. By (46),

$$f_{\delta h} \leq E(\omega_i|x_i) = \int_{-h/2}^{h/2} f_{\varepsilon_i|x_i}(u) du \leq f_{\delta h}. \quad (54)$$

Moreover, define $\xi_\delta = \langle w, \delta \rangle$ such that $E(\xi_\delta^2) = 1$. By (52) and (54),

$$E\{\omega_i \cdot \varphi_{1/(2r)}((w_i, \delta))\} \geq \int_{-h/2}^{h/2} f_{\varepsilon_i|x_i}(u) du \leq f_{\delta h} \cdot \left[1 - E\xi_\delta^2 I(|\xi_\delta| > h/(4r))\right].$$

For $0 < r \leq h/(4n_{1/4})$ with $n_{1/4}$ defined in (47), it follows that

$$\inf_{\delta \in \mathbb{S}^{p-1}} E\{D_0(\delta)\} \geq f_{\delta h} \cdot \left\{1 - \sup_{u \in \mathbb{S}^{p-1}} E\langle w, u \rangle^2 I(|\langle w, u \rangle| \geq n_{1/4})\right\} \geq \frac{3}{4} f_{\delta h}. \quad (55)$$

Turning to the random fluctuation, we will use Theorem 7.3 in Bousquet (2003) (a refined Talagrand’s inequality) to bound

$$\Delta = \sup_{\delta \in \mathbb{S}^{p-1}} \left[D_0^- (\delta) - E\{D_0^-(\delta)\}\right], \quad (56)$$

where $D_0^-(\delta) := -D_0(\delta)$. Note that $0 \leq \varphi_R(u) \leq \min\{(R/2)^2, (R/2)|u|\}$ for all $u \in \mathbb{R}$ and $\omega_i \in \{0, 1\}$. Therefore,

$$0 \leq \chi_i := (\omega_i/h) \cdot \varphi_{1/(2r)}((w_i, \delta)) \leq \omega_i \cdot \min \left\{h/(4r)^2, |\langle w_i, \delta \rangle|/(4r)\right\}.$$
This, combined with (54), yields \( \mathbb{E}(\chi^2) \leq \tilde{f}_h h/(4r)^2 \). We then apply Theorem 7.3 in Bousquet (2003) and obtain that, for any \( t > 0 \),

\[
\Delta \leq \mathbb{E}(\Delta) + \{ \mathbb{E}(\Delta) \}^{1/2} \left( \sqrt{\frac{ht}{4r^2n}} + \sqrt{\frac{\tilde{f}_h h t}{8r^2n}} + \frac{ht}{48r^2n} \right) \leq \frac{5}{4} \mathbb{E}(\Delta) + \sqrt{\frac{\tilde{f}_h h t}{8r^2n}} + \frac{ht}{3r^2n}
\]

with probability at least \( 1 - e^{-t} \), where the second step follows from the inequality that \( ab \leq a^2/4 + b^2 \) for all \( a, b \in \mathbb{R} \).

It remains to bound the expected value \( \mathbb{E}(\Delta) \). Recall that \( \omega_i = 1(|\varepsilon_i| \leq h/2) \in \{0, 1\} \), and hence \( \omega_i \varphi_{h/(2r)}(\langle w, \delta \rangle) = \omega_i^2 \varphi_{h/(2r)}(\langle w, \delta \rangle) = \varphi_{w_1 h/(2r)}(\langle w, \delta \rangle) = \varphi_{h/(2r)}(\langle \omega w, \delta \rangle) \).

Then, we define

\[
z_i = \omega_i w_i \quad \text{and} \quad \mathcal{E}(\delta; z_i) = \varphi_{h/(2r)}(\langle z_i, \delta \rangle), \quad \delta \in \mathbb{S}^{p-1}.
\]

By Rademacher symmetrization,

\[
\mathbb{E}(\Delta) \leq 2\mathbb{E} \left\{ \sup_{\delta \in \mathbb{S}^{p-1}} \frac{1}{nh} \sum_{i=1}^{n} e_i \cdot \mathcal{E}(\delta; z_i) \right\},
\]

where \( e_1, \ldots, e_n \) are independent Rademacher random variables. Since \( \varphi_R(\cdot) \) is \( R \)-Lipschitz, \( \mathcal{E}(\delta; z_i) \) is an \( (h/2r) \)-Lipschitz function in \( \langle z_i, \delta \rangle \), i.e., for any sample \( z_i \) and parameters \( \delta, \delta' \in \mathbb{S}^{p-1} \),

\[
|\mathcal{E}(\delta; z_i) - \mathcal{E}(\delta'; z_i)| \leq \frac{h}{2r} |\langle z_i, \delta \rangle - \langle z_i, \delta' \rangle|.
\]

Moreover, observe that \( \mathcal{E}(\delta; z_i) = 0 \) for any \( \delta \) such that \( \langle z_i, \delta \rangle = 0 \). With the above preparations, we are ready to use Talagrand’s contraction principle to bound \( \mathbb{E}(\Delta) \). Define the subset \( T \subseteq \mathbb{R}^n \) as

\[
T = \left\{ t = (t_1, \ldots, t_n)^T : t_i = \langle z_i, \delta \rangle, i = 1, \ldots, n, \delta \in \mathbb{S}^{p-1} \right\},
\]

and contractions \( \phi_i : \mathbb{R} \to \mathbb{R} \) as \( \phi_i(t) = (2r/h) \cdot \varphi_{h/(2r)}(t) \). By (58), \( |\phi(t) - \phi(s)| \leq |t - s| \) for all \( t, s \in \mathbb{R} \). Applying Talagrand’s contraction principle (see, e.g., Theorem 4.12 and (4.20) in Ledoux and Talagrand (1991)), we have

\[
\mathbb{E}(\Delta) \leq 2\mathbb{E} \left\{ \sup_{\delta \in \mathbb{S}^{p-1}} \frac{1}{nh} \sum_{i=1}^{n} e_i \cdot \mathcal{E}(\delta; z_i) \right\} = \frac{1}{r} \mathbb{E} \left\{ \sup_{t \in T} \frac{1}{n} \sum_{i=1}^{n} e_i \phi_i(t_i) \right\} \leq \frac{1}{r} \mathbb{E} \left( \sup_{t \in T} \frac{1}{n} \sum_{i=1}^{n} e_i t_i \right)
\]

\[
= \frac{1}{r} \mathbb{E} \left( \sup_{\delta \in \mathbb{S}^{p-1}} \frac{1}{n} \sum_{i=1}^{n} e_i \langle z_i, \delta \rangle \right) \leq \frac{1}{r} \mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^{n} e_i z_i \right\|_2 \leq \frac{\tilde{f}_h}{r^2 n} \sqrt{\frac{hp}{r^2 n}}.
\]

This, combined with (56) and (57), yields

\[
\Delta \leq \tilde{f}_h^{1/2} \left( \frac{5}{4} \sqrt{\frac{hp}{r^2 n}} + \sqrt{\frac{ht}{8r^2 n}} \right) + \frac{ht}{3r^2 n}
\]

with probability at least \( 1 - e^{-t} \).

Finally, combining (50), (53), (55) and (59) proves (48). \( \square \)
B.3 Proof of Theorem 2

We keep the notation used in the proof of Theorem 1, and for any \( t \geq 0 \), let \( r = r(n,p,t) \) \( \propto \sqrt{(p+t)/n + h^2} > 0 \) be such that \( \mathbb{P}\{\beta_h \in \beta^* + \Theta(r)\} \geq 1 - 2e^{-t} \), provided \( \sqrt{(p+t)/n} \lesssim h \lesssim 1 \). Define the vector-valued random process

\[
\Delta(\beta) = \Sigma^{-1/2}\{\nabla \hat{Q}_h(\beta) - \nabla \hat{Q}_h(\beta^*) - D(\beta - \beta^*)\},
\]

where \( D = \mathbb{E}\{f_{\epsilon|x}(0)xx^T\} \). Since \( \hat{\beta}_h \) falls in a local neighborhood of \( \beta^* \) with high probability, it suffices to bound the local fluctuation \( \sup_{\beta \in \beta^* + \Theta(r)} \|\Delta(\beta)\|_2 \). By the triangle inequality,

\[
\sup_{\beta \in \beta^* + \Theta(r)} \|\Delta(\beta)\|_2 \leq \sup_{\beta \in \beta^* + \Theta(r)} \|\Delta \|_2 + \sup_{\beta \in \beta^* + \Theta(r)} \|\Delta - \mathbb{E}\Delta(\beta)\|_2 := I_1 + I_2.
\]

We now provide upper bounds for \( I_1 \) and \( I_2 \), respectively.

**Upper Bound for \( I_1 \):** By the mean value theorem for vector-valued functions,

\[
\mathbb{E}\Delta(\beta) = \Sigma^{-1/2}\left(\int_0^1 \nabla^2 Q_h((1-t)\beta^* + t\beta)dt, \beta - \beta^*\right) - \Sigma^{-1/2}D(\beta - \beta^*)
\]

\[
= \Sigma^{-1/2}\left(\int_0^1 \nabla^2 Q_h((1-t)\beta^* + t\beta)dt \sum^{-1/2} - D_0, \Sigma^{1/2}(\beta - \beta^*)\right),
\]

where \( D_0 := \Sigma^{-1/2}D\Sigma^{-1/2} = \mathbb{E}\{f_{\epsilon|x}(0)ww^T\} \). By law of iterative expectation and by a change of variable,

\[
\Sigma^{-1/2}\nabla^2 Q_h(\beta)\Sigma^{-1/2} = \mathbb{E}\{K_h(\langle x, \beta \rangle - y)ww^T\} = \mathbb{E}\left\{\int_{-\infty}^{\infty} K(u)f_{\epsilon|x}(\langle x, \beta^* \rangle - hu)du \cdot ww^T\right\}.
\]

For notational convenience, let \( v = \Sigma^{-1/2}(\beta - \beta^*) \) with \( \beta \in \beta^* + \Theta(r) \), so that \( \|v\|_2 \leq r \) and

\[
\nabla^2 Q_h((1-t)\beta^* + t\beta) = \mathbb{E}\left\{\int_{-\infty}^{\infty} K(u)f_{\epsilon|x}(t\langle w, v \rangle - hu)du \cdot ww^T\right\}.
\]

By the Lipschitz continuity of \( f_{\epsilon|x}(\cdot) \), i.e. \( |f_{\epsilon|x}(u) - f_{\epsilon|x}(0)| \leq l_0|u| \) for all \( u \in \mathbb{R} \) almost surely for \( x \), we have

\[
\|\Sigma^{-1/2}\nabla^2 Q_h((1-t)\beta^* + t\beta)\Sigma^{-1/2} - D_0\|_2
\]

\[
= \mathbb{E}\left\|\int_{-\infty}^{\infty} K(u)f_{\epsilon|x}(t\langle w, v \rangle - hu)du \cdot ww^T\right\|_2
\]

\[
\leq l_0t \sup_{\|u\|_2 = 1} \mathbb{E}\|\langle w, u \rangle^2\|_2 + l_0\kappa_1h \sup_{\|u\|_2 = 1} \mathbb{E}\|\langle w, u \rangle^2\|
\]

\[
\leq l_0t \left(\sup_{\|u\|_2 = 1} \mathbb{E}\|\langle w, u \rangle^2\|\right)^{2/3} \left(\mathbb{E}\|\langle w, v \rangle^3\right)^{1/3} + l_0\kappa_1h \leq l_0(m_3rt + \kappa_1h),
\]

where the third inequality holds by the Cauchy-Schwarz inequality. Consequently,

\[
\sup_{\beta \in \beta^* + \Theta(r)} \|\mathbb{E}\Delta(\beta)\|_2 \leq l_0(0.5m_3r + \kappa_1h) \cdot r.
\]
Upper bound for $I_2$: Next, we provide an upper bound for $\Delta(\beta) - \mathbb{E}\Delta(\beta)$. Define the centered gradient process $G(\beta) = \Sigma^{-1/2}\{\nabla \hat{Q}_h(\beta) - \nabla Q_h(\beta)\}$, so that $\Delta(\beta) - \mathbb{E}\Delta(\beta) = G(\beta) - G(\beta^*)$. Again, by a change of variable $v = \Sigma^{1/2}(\beta - \beta^*)$, we have

$$\sup_{\beta \in \beta^* + \Theta(\varepsilon)} \|\Delta(\beta) - \mathbb{E}\Delta(\beta)\|_2 \leq \sup_{\beta \in \beta^* + \Theta(\varepsilon)} \|G(\beta) - G(\beta^*)\|_2 = \sup_{\|v\|_2 \leq r} \|G(\beta^* + \Sigma^{-1/2}v) - G(\beta^*)\|_2,$$

where $\Delta_0(\cdot)$ defined above satisfies $\Delta_0(0) = 0$ and $\mathbb{E}\{\Delta_0(v)\} = 0$. Taking the gradient with respect to $v$ yields

$$\nabla \Delta_0(v) = \frac{1}{n} \sum_{i=1}^{n} \{K_{i,v}^T w^i - \mathbb{E}(K_{i,v}^T w^i)\},$$

where $K_{i,v} := K_h((w_i, v) - \varepsilon_i)$ satisfies $0 \leq K_{i,v} \leq \kappa_v h^{-1}$. For any $u, v \in \mathbb{S}^{p-1}$ and $\lambda \in \mathbb{R}$, using the elementary inequality $|e^u - 1 - u| \leq u^2 e^{|u|}/2$, we obtain

$$\mathbb{E}\left\{\exp\left\{\lambda n^{1/2}\langle u, \nabla \Delta_0(v) v \rangle\right\}\right\}
\leq \prod_{i=1}^{n} \left\{1 + \frac{\lambda^2}{2n} \frac{f_{\hat{t}}(|u|)}{\sqrt{\mathbb{E}\{\langle w_i, u \rangle \langle w_i, v \rangle\}}} \mathbb{E}\{K_{i,v} \langle w_i, u \rangle \langle w_i, v \rangle - \mathbb{E}(K_{i,v} \langle w_i, u \rangle \langle w_i, v \rangle)\}^2 e^{\frac{n\lambda^2}{2\sqrt{\mathbb{E}\{\langle w_i, u \rangle \langle w_i, v \rangle\}^2}}}ight\},$$

where we use the bound $\mathbb{E}\{\langle w_i, u \rangle \langle w_i, v \rangle\} \leq (\mathbb{E}\{\langle w_i, u \rangle^2\})^{1/2}(\mathbb{E}\{\langle w_i, v \rangle^2\})^{1/2} = 1$ in the second inequality. Moreover, the first and second conditional moments of $K_{i,v}$ can be rewritten as follows:

$$\mathbb{E}(K_{i,v}|x_i) = \frac{1}{h} \int_{-\infty}^{\infty} K\left(\frac{\langle w_i, v \rangle - t}{h}\right)f_{\hat{t}}(\tau)|x_i\rangle\langle t|dt = \int_{-\infty}^{\infty} K(u)f_{\hat{t}}(\tau)|x_i\rangle\langle \langle w_i, v \rangle - hu|du;$$

$$\mathbb{E}(K_{i,v}^2|x_i) = \frac{1}{h^2} \int_{-\infty}^{\infty} K^2\left(\frac{\langle w_i, v \rangle - t}{h}\right)f_{\hat{t}}(\tau)|x_i\rangle\langle t|dt = \frac{1}{h} \int_{-\infty}^{\infty} K^2(u)f_{\hat{t}}(\tau)|x_i\rangle\langle \langle w_i, v \rangle - hu|du,$$

from which it follows that $|\mathbb{E}(K_{i,v}|x_i)| \leq \tilde{f}$ and $\mathbb{E}(K_{i,v}^2|x_i) \leq \kappa_v \tilde{f} h^{-1}$ almost surely.

By the Cauchy-Schwarz inequality and the inequality $ab \leq a^2/2 + b^2/2$, $a, b \in \mathbb{R}$, we have

$$\mathbb{E}\{\langle w_i, u \rangle \langle w_i, v \rangle\}^2 e^{t\langle w_i, u \rangle \langle w_i, v \rangle} \leq \mathbb{E}(\langle w_i, u \rangle^2)^{1/2}(\mathbb{E}(\langle w_i, v \rangle^2)^{1/2}, \text{ valid for any } t > 0.$$


Given a unit vector $u$, let $\chi = (w, u)^2/(2v_1)^2$ so that under Condition 3.5, $\mathbb{P}(\chi \geq u) \leq 2e^{-2u}$ for any $u \geq 0$. It follows that $\mathbb{E}(e^\chi) = 1 + \int_0^\infty e^u e^{\chi} \mathbb{P}(\chi \geq u)du \leq 1 + 2 \int_0^\infty e^u du = 3$, and
\[
\mathbb{E}(\chi^2 e^\chi) = \int_0^\infty (u^2 + 2u)e^u \mathbb{P}(\chi \geq u)du \leq 2 \int_0^\infty (u^2 + 2u)e^{-u}du = 8.
\]
Taking the supremum over $u \in \mathbb{S}^{p-1}$, we have
\[
\sup_{u \in \mathbb{S}^{p-1}} \mathbb{E}e^{(w, u)^2/(2v_1)^2} \leq 3 \quad \text{and} \quad \sup_{u \in \mathbb{S}^{p-1}} \mathbb{E}(w, u)^4 e^{(w, u)^2/(2v_1)^2} \leq 8(2v_1)^4.
\]
Substituting the above bounds into (63) yields that, for any $|\lambda| \leq \min\{n^{1/2}h/(4\kappa_u v_1^2), n^{1/2}/\bar{f}\}$,
\[
\mathbb{E}\exp\left\{\lambda n^{1/2}e<(w, \nabla \Delta_0(v)v)>\right\}
\leq \prod_{i=1}^n \left[1 + \frac{e\lambda^2}{2n} \mathbb{E}\left(K_{i,v}(w_i, u)(w_i, v) - \mathbb{E}(K_{i,v}(w_i, u)(w_i, v))\right)^2 e^{(w_i, u)(w_i, v)/(4v_1^2)}\right]
\leq \prod_{i=1}^n \left[1 + \frac{e\lambda^2}{n} \mathbb{E}(K_{i,v}(w_i, u)(w_i, v))^2 e^{(w_i, u)(w_i, v)/(4v_1^2)} + \frac{e\lambda^2}{n} \mathbb{E}(K_{i,v}(w_i, u)(w_i, v))^2 \mathbb{E}e^{(w_i, u)(w_i, v)/(4v_1^2)}\right]
\leq \prod_{i=1}^n \left(1 + C_0^2 \lambda^2/nh\right) \leq \exp\left\{C_0^2 \lambda^2/(2h)\right\},
\]
where $C_0 > 0$ depends only on $(v_1, \kappa_u, \bar{f})$. We have thus verified condition (A.4) in Spokoiny (2013) with $g = \min\{h/(4\kappa_u v_1^2), 1/\bar{f}\}/(n/2)^{1/2}$ and $\nu_0 = C_0 h^{-1/2}$. Applying Theorem A.3 therein, we obtain that with probability at least $1 - e^{-t}$,
\[
\sup_{\|v\|_2 \leq r} \|\Delta_0(v)\|_2 \leq 6C_0r\sqrt{\frac{4p + 2t}{nh}}
\]
as long as $h \geq 8\kappa_u v_1^2\sqrt{(2p + t)/n}$ and $n \geq 4\bar{f}^2(2p + t)$.

Together with (61) and (62), this implies that with probability at least $1 - e^{-t}$,
\[
\sup_{\beta \in \beta^* + \Theta(r)} \|\Delta(\beta)\|_2 \leq 6C_0r\sqrt{\frac{4p + 2t}{nh}} + l_0(0.5m_3 r + \kappa_1 h) r.
\]
Recall from the beginning of the proof that $\hat{\beta}_h \in \beta^* + \Theta(r)$ with probability at least $1 - 2e^{-t}$ with $r = r(n, p, t) \asymp \sqrt{(p + t)/n + h^2}$. Combined with (64), we conclude that with probability at least $1 - 3e^{-t}$, $\|\Delta(\hat{\beta}_h)\|_2 \lesssim (p + t)/(h^{1/2}n) + h\sqrt{(p + t)/n + h},$ as claimed. \hfill \Box

**B.4 Proof of Theorem 3**

Let $\mathbf{a} \in \mathbb{R}^p$ be an arbitrary vector defining a linear functional of interest. Given $h = h_n > 0$, define $S_n = n^{-1/2}\sum_{i=1}^n \gamma_i \xi_i$ and its centered version $S_n^0 = S_n - \mathbb{E}(S_n)$, where $\xi_i = \tau - \mathbb{R}(-\varepsilon_i/h)$ and $\gamma_i = \langle D^{-1}\mathbf{a}, \mathbf{x}_i \rangle$. By the Lipschitz continuity of $f_{\varepsilon|\mathbf{x}}$ around 0 and
the fundamental theorem of calculus, it can be shown that $|E(\xi|x_i)| \leq 0.5 l_0 \kappa_2 h^2$, from which it follows by the law of iterated expectation that $|E(\gamma_i \xi_i)| \leq 0.5 l_0 \kappa_2 ||D^{-1}a||_p \cdot h^2$.

Let $\delta_n = (p + \log n)/n$. Then, applying (60) and (64) with $t = \log n$ and the triangle inequality, we obtain that under the constraint $\delta_n^{1/2} \lesssim h \lesssim 1$,

$$\left| n^{1/2} \langle a, \hat{\beta}_h - \beta^* \rangle - S_n^0 \right|$$

$$= n^{1/2} \left| \left( \Sigma^{1/2} D^{-1} a, \Sigma^{-1/2} D (\hat{\beta}_h - \beta^*) - \Sigma^{-1/2} \frac{1}{n} \sum_{i=1}^n \left\{ \tau - K(-\epsilon_i/h) \right\} x_i \right) \right| + |E(S_n)|$$

$$\leq c_1 ||D^{-1}a||_p \cdot n^{1/2} (h^{-1/2} \delta_n + h^2)$$

(65)

with probability at least $1 - 3n^{-1}$ for some constant $c_1 > 0$.

For the centered partial sum $S_n^0 = S_n - E(S_n) = n^{-1/2} \sum_{i=1}^n (1 - E) \gamma_i \xi_i$, we have $\text{var}(S_n^0) = \text{var}(S_n) = E(\gamma_i^2) - \{E(\gamma_i)\}^2$, where $\gamma = \langle D^{-1}a, x \rangle$ and $\xi = \tau - K(-\epsilon/h)$. By the Berry-Esseen inequality (see, e.g., Tyurin (2011)),

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{ S_n^0 \leq \text{var}(S_n)^{1/2} x \right\} - \Phi(x) \right| \leq \frac{\mathbb{E}\{|\gamma \xi - E(\gamma \xi)|^3\}}{2\mathbb{E}\{\gamma \xi^2\}^{3/2} \sqrt{n}}.$$  \hspace{1cm} (66)

We have shown that $|E(\gamma \xi)| \lesssim ||D^{-1}a||_p \cdot h^2$. Following a similar argument in the proof of Lemma 1, we have $E(\xi^2|x) \lesssim (1 - \tau) + h^2$. For $h$ sufficiently small, $\text{var}(S_n) = \{\tau(1 - \tau) + O(h)\} ||D^{-1}a||_p^2$ and $\mathbb{E}(\gamma \xi^3) \leq \max(\tau, 1 - \tau) \mathbb{E}(\xi^2) |\gamma|^3 \leq m_3 \{\tau(1 - \tau) + O(h^2)\} ||D^{-1}a||_p^2$.

Substituting these bounds into (66) yields

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{ S_n^0 \leq \text{var}(S_n)^{1/2} x \right\} - \Phi(x) \right| \leq c_2 n^{-1/2}$$

(67)

for some constant $c_2 > 0$. Set $\sigma^2 = \tau(1 - \tau) ||D^{-1}a||_p^2$, and we have shown that $|\text{var}(S_n)/\sigma^2 - 1| \lesssim h$. By an application of Lemma A.7 in the supplement of Spokoiny and Zhilova (2015), for sufficiently small $h$, we have

$$\sup_{x \in \mathbb{R}} \left| \Phi(x/\text{var}(S_n)^{1/2}) - \Phi(x/\sigma) \right| \leq c_3 h.$$  \hspace{1cm} (68)

Before proceeding, we note that the constants $c_1$–$c_3$ appeared above are all independent of $a$. Let $G \sim N(0, 1)$. Putting together the above derivations, for any $x \in \mathbb{R}$ and $a \in \mathbb{R}^p$, we obtain

$$\mathbb{P}\left( n^{1/2} \langle a, \hat{\beta}_h - \beta^* \rangle \leq x \right)$$

$$\leq \mathbb{P}\left\{ S_n^0 \leq x + c_1 ||D^{-1}a||_p \cdot n^{1/2} (h^{-1/2} \delta_n + h^2) \right\} + 3n^{-1}$$

$$\leq \mathbb{P}\left\{ \text{var}(S_n)^{1/2} G \leq x + c_1 ||D^{-1}a||_p \cdot n^{1/2} (h^{-1/2} \delta_n + h^2) \right\} + c_2 n^{-1/2} + 3n^{-1}$$

$$\leq \mathbb{P}\{ \sigma G \leq x + c_1 ||D^{-1}a||_p \cdot n^{1/2} (h^{-1/2} \delta_n + h^2) \} + c_2 n^{-1/2} + c_3 h + 3n^{-1}$$

$$\leq \mathbb{P}\{ \sigma \leq x + c_1 (2\pi)^{-1/2} ||D^{-1}a||_p \cdot n^{1/2} (h^{-1/2} \delta_n + h^2) / \sigma \} + c_2 n^{-1/2} + c_3 h + 3n^{-1},$$

where the first, second, and third inequalities holds by (65), (67), and (68), respectively, and the last inequality follows from the fact that for any $a \leq b$, $\Phi(b/\sigma) - \Phi(a/\sigma) \leq (2\pi)^{-1/2} (b - a)/\sigma$. A similar argument leads to a series of reverse inequalities. Note
the above bounds are independent of $x$ and $a$, and therefore hold uniformly over all $x$ and $a$. Putting together the pieces, we conclude that under the bandwidth requirement $\delta_{n}^{1/2} \lesssim h \lesssim 1$,

$$\sup_{x \in \mathbb{R}, a \in \mathbb{R}^{p}} \mathbb{P}(n^{1/2}(a, \hat{\beta}_{h} - \beta^{*}) \leq \sigma_{x} x) - \Phi(x) \lesssim \frac{p + \log n}{(nh)^{1/2}} + n^{1/2}h^{2},$$

as claimed.  

B.5 Proof of Theorem 4

Keep the notation used in the proof of Theorem 1. With nonnegative weights $w_{i}$, the corresponding weighted loss $\hat{Q}_{h}^{i}$ in (9) is convex, and thus the first-order condition $\nabla \hat{Q}_{h}(\beta^{*}) = 0$ holds. We use the same localized argument as in the proof of Theorem 1. For some $r > 0$ to be determined, define $\hat{\beta}_{h} = \beta^{*} + \eta(\hat{\beta}_{h}^{*} - \beta^{*})$, where $\eta := \sup\{u \in [0, 1] : u(\hat{\beta}_{h}^{*} - \beta^{*}) \in \Theta(r)\}$. Similar to (44), we have

$$\|\hat{\beta}_{h} - \beta^{*}\|_{\Sigma} \leq \frac{\|\nabla \hat{Q}_{h}(\beta^{*})\|_{\Omega}}{\inf_{\beta \in \beta + \Theta(r)} D^{\ast}(\beta, \beta^{*})/\|\beta - \beta^{*}\|_{\Sigma}},$$

(69)

where $D^{\ast} : \mathbb{R}^{p} \times \mathbb{R}^{p} \to \mathbb{R}^{+}$ denotes the symmetrized Bregman divergence associated with $\hat{Q}_{h}^{i}$. Next, we present two lemmas on the upper and lower bounds of the numerator and denominator of (69), respectively.

Lemma 3. Assume that Conditions 3.1, 3.2, and 3.5 hold. For any $t \geq 0$, there exists some event $\mathcal{E}_{1}(t)$ with $\mathbb{P}\{\mathcal{E}_{1}(t)\} \geq 1 - 2e^{-t}$ such that conditioned on $\mathcal{E}_{1}(t)$, the bound (45) holds, and with $\mathbb{P}^{*}$-probability at least 1 $- e^{-t}$,

$$\|\nabla \hat{Q}_{h}(\beta^{*})\|_{\Omega} \leq C_{1}\left(\sqrt{\frac{p + t}{n}} + h^{2}\right)$$

(70)

as long as $n \gtrsim p + t$, where $C_{1} > 0$ is a constant depending only on $(\upsilon_{1}, \kappa_{2}, l_{0})$ and $\tau$.

Lemma 4. Let $r = h/(4\eta_{1/4})$ with $\eta_{1/4}$ defined in (47), and $t \geq 0$. Suppose the sample size and bandwidth satisfy $(p + \log n + t)/n \lesssim h \lesssim 1$. Then, there exists some event $\mathcal{E}_{2}(t)$ with $\mathbb{P}\{\mathcal{E}_{2}(t)\} \geq 1 - 3e^{-t}$ such that, conditioning on $\mathcal{E}_{2}(t)$,

$$\inf_{\beta \in \beta^{*} + \Theta(r)} \frac{D(\beta, \beta^{*})}{\|\beta - \beta^{*}\|_{\Sigma}} \geq 0.5\kappa_{l_{0}}$$

and $\mathbb{P}^{*}\left\{\inf_{\beta \in \beta^{*} + \Theta(r)} \frac{D^{\ast}(\beta, \beta^{*})}{\|\beta - \beta^{*}\|_{\Sigma}} \geq 0.5\kappa_{l_{0}}\right\} \geq 1 - e^{-t}.$

(71)

Recall from the proof of Theorem 1 that the upper bound on $\|\hat{\beta}_{h} - \beta^{*}\|_{\Sigma}$ depends on $\|\nabla \hat{Q}_{h}(\beta^{*})\|_{\Omega}$ and $\inf_{\beta \in \beta^{*} + \Theta(r)} \{D(\beta, \beta^{*})/\|\beta - \beta^{*}\|_{\Sigma}^{2}\}$. In view of Lemmas 3 and 4, the bound (16) holds whenever the event $\mathcal{E}(t) := \mathcal{E}_{1}(t) \cap \mathcal{E}_{2}(t)$ occurs. Putting the bounds (69), (70) and (71) together, we conclude that with $\mathbb{P}^{*}$-probability at least $1 - 2e^{-t}$ conditioned on $\mathcal{E}(t)$,

$$\|\hat{\beta}_{h} - \beta^{*}\|_{\Sigma} \leq 2(\kappa_{l_{0}})^{-1}C_{1}\left(\sqrt{\frac{p + t}{n}} + h^{2}\right) < r = h/(4\eta_{1/4})$$

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as long as \( \max\{(p + \log n) / n, \sqrt{(p + t) / n}\} \leq h \leq 1 \). Consequently, \( \tilde{\beta}_h^p \) falls into the interior of \( \beta^* + \Theta(r) \), thereby implying \( \eta = 1 \) and \( \tilde{\beta}_h^p = \beta_h^p \). This completes the proof. □

B.5.1 PROOF OF LEMMA 3

By the triangle inequality,

\[
\|\nabla \hat{Q}_h^{\beta^*}\|_\Omega \leq \|\nabla \hat{Q}_h^{\beta^*} - \nabla \hat{Q}_h^{\beta^*}\|_\Omega + \|\nabla \hat{Q}_h^{\beta^*}\|_\Omega := I_1 + I_2. \tag{72}
\]

Lemma 1 provides an upper bound for \( I_2 \) with high probability, and thus it suffices to bound \( I_1 \). To this end, we will employ Hoeffding’s and Bernstein’s inequalities.

Note that \( \Sigma^{-1/2} \nabla \hat{Q}_h^{\beta^*} = (1/n) \sum_{i=1}^n w_i \xi_i \) and \( \mathbb{E}^* \{ \Sigma^{-1/2} \nabla \hat{Q}_h^{\beta^*} \} = \Sigma^{-1/2} \nabla \hat{Q}_h^{\beta^*} \), where \( w_i = \Sigma^{-1/2} x_i \) and \( \xi_i = K(-\varepsilon_i / h) - \tau \) with \( \varepsilon_i = y_i - (x_i, \beta^*) \). Using a similar covering argument as in the proof of Lemma 1, for any \( \epsilon \in (0, 1) \), there exists an \( \epsilon \)-net \( \mathcal{N}_\varepsilon \) of the unit sphere \( \mathbb{S}^{p-1} \) with cardinality \( |\mathcal{N}_\varepsilon| \leq (1 + 2/\epsilon)^p \) such that

\[
\|\nabla \hat{Q}_h^{\beta^*} - \nabla \hat{Q}_h^{\beta^*}\|_\Omega \leq \frac{1}{1 - \epsilon} \max_{u \in \mathcal{N}_\varepsilon} \frac{1}{n} \sum_{i=1}^n e_i : \xi_i(u, w_i),
\]

where \( e_i = w_i - 1 \) are independent Rademacher random variables. By Hoeffding’s inequality,

\[
\mathbb{P}^* \left\{ \frac{1}{n} \sum_{i=1}^n \xi_i(u, w_i) \geq \left( \frac{1}{n} \sum_{i=1}^n \xi_i^2(u, w_i) \right)^{1/2} \sqrt{\frac{2u}{n}} \right\} \leq e^{-u} \text{ for any } u \geq 0.
\]

For the data-dependent quantity \( (1/n) \sum_{i=1}^n \xi_i^2(u, w_i)^2 \), recall from the proof of Lemma 1 that \( |\xi_i| \leq \tau = \max(\tau, 1 - \tau, 1 + \tau) \), \( \mathbb{E}(\xi_i^4|x_i) \leq \nu_i^4 = \tau(1 - \tau) + (1 + \tau) l_0 \kappa_2 h^2 \) and hence \( \mathbb{E}(\xi_i^2(u, w_i)^2) \leq \nu_i^2 \). Combined with the sub-Gaussianity of \( w_i \), we obtain that, for \( k = 2, 3, \ldots \),

\[
\mathbb{E}(\xi_i^2(u, w_i)^2)^k \leq \nu_i^2 \tau^2 (k-1)! v_1^2 k \int_0^\infty \mathbb{P}(\langle u, w_i \rangle) \geq v_1 u) u^{2k-1} du \\
\leq \frac{\nu_i^2 \tau^2 (k-1)! v_1^2}{4k} \int_0^\infty u^{2k-1} e^{-u^2/2} du \\
= \frac{\nu_i^2 \tau^2 (k-1)! v_1^2}{2k} \cdot 2k \int_0^\infty v^{k-1} e^{-v} dv = 2^{k+1} k! \cdot v_1^2 \tau^{2(k-1)} v_1^2.
\]

In particular, \( \mathbb{E}(\xi_i^4(u, w_i)^4) \leq (4v_1^2 \tau) v_1^4 \) and \( \mathbb{E}(\xi_i^2(u, w_i)^2)^k \leq \frac{h_i^2}{2^k} \cdot (4v_1^2 \tau) v_1^4 \cdot (2^2 v_1^2)^{k-2} \) for \( k \geq 3 \). With the above calculations, it follows from Bernstein’s inequality that

\[
\mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n \xi_i^2(u, w_i)^2 \geq v_1^2 + 4v_1^2 v_1^2 \sqrt{\frac{2v}{n} + 2 v_1^2 \tau^2 v_1^2} \right\} \leq e^{-v} \text{ for any } v \geq 0.
\]

In the above analysis, we set \( \epsilon = 2/(e^2 - 1) \) so that \( (1 + 2/\epsilon)^p = e^{2p} \), and take \( u = v = 2p + t \). Applying the union bound, we conclude that

\[
\max_{w \in \mathcal{N}_\varepsilon} \frac{1}{n} \sum_{i=1}^n \xi_i^2(u, w_i)^2 \leq v_1^2 + 4v_1^2 v_1^2 \sqrt{\frac{4p + 2t}{n} + 2 v_1^2 \tau^2 2p + t} \tag{73}
\]

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with probability at least $1 - e^{-t}$. Let $\mathcal{E}_1(t)$ be the event that (73) and (45) hold, so that $\mathbb{P}\{\mathcal{E}_1(t)\} \geq 1 - 2e^{-t}$. Using the union bound again, we obtain that with $\mathbb{P}^*$-probability at least $1 - e^{-t}$ conditioned on $\mathcal{E}_1(t)$,

$$\|\nabla \widehat{Q}_n^* - \nabla \widehat{Q}_h^*(\beta^*)\|_\Omega \leq C \sqrt{\frac{p + t}{n}}$$

as long as $n \gtrsim p + t$, where $C > 0$ depends only on $(v_1, \kappa_2, l_0)$ and $\tau$. This, together with (72) and (45), proves the claimed bound. \hfill \Box

### B.5.2 Proof of Lemma 4

Using arguments similar to (50), (51), and (53) in the proof of Lemma 2, we obtain

$$D^\delta(\beta, \beta^*) = \frac{1}{n} \sum_{i=1}^{n} \left\{ \mathcal{R} \left( \frac{(x_i, \beta) - y_i}{h} \right) - \mathcal{R} \left( \frac{-\varepsilon_i}{h} \right) \right\} w_i \langle x_i, \beta - \beta^* \rangle$$

$$\geq \kappa_t \|\beta - \beta^*\|_{\Sigma}^2 \cdot \frac{1}{nh} \sum_{i=1}^{n} (1 + e_i) \omega_i \cdot \varphi_{h/2r}(\langle w_i, \delta \rangle),$$

where $\delta = \Sigma^{1/2}(\beta - \beta^*)/\|\beta - \beta^*\|_{\Sigma}$, $\omega_i = 1(|\varepsilon_i| \leq h/2)$, and $\varphi(\cdot)$ is as in (52). Recall from (53) the definition of $D_0(\delta)$. We have

$$\inf_{\beta \in \beta^* + \Theta(r)} \frac{D^\delta(\beta, \beta^*)}{\kappa_t \|\beta - \beta^*\|_{\Sigma}} \geq \inf_{\delta \in \mathbb{S}^{p-1}} D_0(\delta) - \sup_{\delta \in \mathbb{S}^{p-1}} \{ D_0(\delta) - D_0^\delta(\delta) \}. \quad (74)$$

In what to follow, we obtain lower and upper bounds for $D_0(\delta)$ and $\sup_{\delta \in \mathbb{S}^{p-1}} \{ D_0(\delta) - D_0^\delta(\delta) \}$, respectively. For the latter, we write

$$\Gamma_n = \Gamma_n(e_1, \ldots, e_n) = \sup_{\delta \in \mathbb{S}^{p-1}} \{ D_0(\delta) - D_0^\delta(\delta) \} = \sup_{\delta \in \mathbb{S}^{p-1}} \frac{1}{nh} \sum_{i=1}^{n} e_i \omega_i \cdot \varphi_{h/2r}(\langle w_i, \delta \rangle).$$

Since $\varphi_R(u) \leq (R/2)^2$ and $e_i \in \{-1, 1\}$, we have $\mathbb{E}^*\{ e_i \omega_i \cdot \varphi_{h/2r}(\langle w_i, \delta \rangle) \}^2 \leq (h/4r)^4 \omega_i$ and $|e_i \omega_i \cdot \varphi_{h/2r}(\langle w_i, \delta \rangle)| \leq (h/4r)^2$. Then, by Bousquet’s version of Talagrand’s inequality with explicit constants (see Theorem 7.3 in Bousquet (2003)), for every $t \geq 0$,

$$\Gamma_n \leq \mathbb{E}^*(\Gamma_n) + \sqrt{\frac{h^2}{(4r)^4} \mathbb{E}^*(\Gamma_n) \mathbb{E}^*(\Gamma_n) + 4 \mathbb{E}^*(\Gamma_n) \frac{h}{(4r)^2} \mathbb{E}^*(\Gamma_n) + \frac{h}{(4r)^2} \frac{t}{3n}}$$

$$\leq 2 \mathbb{E}^*(\Gamma_n) + \frac{h}{(4r)^2} \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} \omega_i \right)^{1/2} \sqrt{\frac{2t}{n} + \frac{4t}{3n}} \right\}.$$
with probability at least $1 - e^{-t}$. Further, by the Lipschitz continuity of $u \to \varphi_R(u)$ and Talagrand’s contraction principle,

$$
\mathbb{E}^*(\Gamma_n) \leq \frac{1}{2r} \mathbb{E}^* \left\{ \sup_{\delta \in \mathbb{S}^{p-1}} \frac{1}{n} \sum_{i=1}^{n} e_i(\omega_i \mathbf{w}_i, \delta) \right\} \\
\leq \frac{1}{2r} \mathbb{E}^* \left\| \frac{1}{n} \sum_{i=1}^{n} e_i \omega_i \mathbf{w}_i \right\|_2 \\
\leq \frac{1}{2rn} \left( \sum_{i=1}^{n} \omega_i \left\| \mathbf{w}_i \right\|_2^2 \right)^{1/2} \\
\leq \max_{1 \leq i \leq n} \left\| \mathbf{w}_i \right\|_2 \cdot \left( \frac{1}{n} \sum_{i=1}^{n} \omega_i \right)^{1/2} \cdot \frac{1}{2rn^{1/2}}.
$$

Together, the last two displays imply

$$
\Gamma_n \leq \left( \frac{1}{n} \sum_{i=1}^{n} \omega_i \right)^{1/2} \left\{ \max_{1 \leq i \leq n} \left\| \mathbf{w}_i \right\|_2 \frac{1}{rn^{1/2}} + \frac{h}{(4r)^2} \sqrt{\frac{2t}{n}} \right\} + \frac{h}{(4r)^2} \frac{4t}{3n} \tag{75}
$$

with $\mathbb{P}^*$-probability at least $1 - e^{-t}$.

Next we provide upper bounds for the data-dependent quantities $\max_{1 \leq i \leq n} \left\| \mathbf{w}_i \right\|_2$ and $(1/n) \sum_{i=1}^{n} \omega_i$. As in the proof of Lemma 3, for any $\epsilon \in (0, 1)$, there exits a subset $\mathcal{N}_\epsilon \subseteq \mathbb{S}^{p-1}$ with $|\mathcal{N}_\epsilon| \leq (1 + 2/\epsilon)^p$ such that $\max_{1 \leq i \leq n} \left\| \mathbf{w}_i \right\|_2 \leq (1 - \epsilon)^{-1} \max_{1 \leq i \leq n} \max_{u \in \mathcal{N}_\epsilon} \langle \mathbf{u}, \mathbf{w}_i \rangle$. Given $1 \leq i \leq n$ and $\mathbf{u} \in \mathcal{N}_\epsilon$, Condition 3.5 indicates $\mathbb{P}(\|\langle \mathbf{u}, \mathbf{w}_i \rangle\| \geq v_1 u) \leq 2e^{-u^2/2}$ for any $u \in \mathbb{R}$. Taking the union bound over $i$ and $\mathbf{u}$, and setting $u = \sqrt{2t + 2 \log(2n) + 2p \log(1 + 2/\epsilon)}$ ($t > 0$), we obtain that with probability at least $1 - 2n(1 + 2/\epsilon)^p e^{-u^2/2} = 1 - e^{-t}$, $\max_{1 \leq i \leq n} \left\| \mathbf{w}_i \right\|_2 \leq (1 - \epsilon)^{-1} v_1 \sqrt{2t + 2 \log(2n) + 2p \log(1 + 2/\epsilon)}$. Minimizing this upper bound with respect to $\epsilon \in (0, 1)$, we conclude that for any $t > 0$, the event

$$
\mathcal{E}_{\text{max}}(t) := \left\{ \max_{1 \leq i \leq n} \left\| \mathbf{w}_i \right\|_2^2 \leq 2v_1^2 \left\{ 3.7p + \log(2n) + t \right\} \right\} \tag{76}
$$

occurs with probability at least $1 - e^{-t}$. For $(1/n) \sum_{i=1}^{n} \omega_i = (1/n) \sum_{i=1}^{n} 1(|\varepsilon_i| \leq h/2)$, note that $\mathbb{E}\omega_i \leq f h$, and by Bernstein’s inequality,

$$
\frac{1}{n} \sum_{i=1}^{n} \omega_i \leq \mathbb{E}\omega_i + \sqrt{\mathbb{E}\omega_i \cdot \frac{2t}{n}} + \frac{t}{3n} \leq \left( \sqrt{\mathbb{E}\omega_i} + \sqrt{\frac{t}{2n}} \right)^2
$$

with probability greater than $1 - e^{-t}$. Let be $\mathcal{E}_{\text{smo}}(t)$ the event that this bound holds, where “smo” stands for “smoothing”. Putting together the pieces, we conclude that conditioned on $\mathcal{E}_{\text{max}}(t) \cap \mathcal{E}_{\text{smo}}(t)$,

$$
\sup_{\delta \in \mathbb{S}^{p-1}} \{ D_0(\delta) - D_0^v(\delta) \} \\
\leq \{(f h)^{1/2} + (0.5t/n)^{1/2}\} \left\{ v_1 (\sqrt{7.4p + 2 \log(2n) + 2t})^{1/2} \frac{h}{(4r)^2} \sqrt{\frac{2t}{n}} + \frac{h}{(4r)^2} \sqrt{\frac{4t}{3n}} \right\} \tag{77}
$$
holds with \( \mathbb{P} \)-probability greater than \( 1 - e^{-t} \).

Turning to \( D_0(\delta) \), let \( \mathcal{E}_{\text{loc}}(t) \) be the event that the bound (48) with \( r = h/(4n/4) \) holds, implying local strong convexity. By Lemma 2, \( \mathbb{P}\{\mathcal{E}_{\text{loc}}(t)\} \geq 1 - e^{-t} \). Provided that \( (p + \log n + t)/n \lesssim h \lesssim 1 \), substituting (77) and (48) into (74) yields

\[
\inf_{\beta \in \beta^* + \Theta(r)} \mathbb{E}\{\mathcal{D}(\beta, \beta^*) / \mathbb{E}\{\mathcal{D}(\beta, \beta^*)/2\}
\]

with \( \mathbb{P} \)-probability greater than \( 1 - e^{-t} \) conditioned on \( \mathcal{E}_{\text{max}}(t) \cap \mathcal{E}_{\text{smo}}(t) \cap \mathcal{E}_{\text{loc}}(t) \). This completes the proof of (71) by setting \( \mathcal{E}_2(t) = \mathcal{E}_{\text{max}}(t) \cap \mathcal{E}_{\text{smo}}(t) \cap \mathcal{E}_{\text{loc}}(t) \).

\( \square \)

B.6 Proof of Theorem 5

The proof is based on an argument similar to that used in the proof of Theorem 2. To begin with, define the random process

\[
\Delta^b(\beta) = \Sigma^{-1/2}\{\nabla \hat{Q}_h^b(\beta) - \nabla \hat{Q}_h^b(\beta^*) - D(\beta - \beta^*)\}, \quad \beta \in \mathbb{R}^p.
\]

(78)

For a prespecified \( r > 0 \), a key step is to bound the local fluctuation \( \sup_{\beta \in \beta^* + \Theta(r)} \|\Delta^b(\beta)\|_2 \). Since \( \mathbb{E}\{w_i\} = 1 \), we have \( \mathbb{E}^*\{\nabla \hat{Q}_h^b(\beta)\} = \nabla \hat{Q}_h(\beta) \). Define the (conditionally) centered process

\[
G^b(\beta) = \Sigma^{-1/2}\{\nabla \hat{Q}_h^b(\beta) - \nabla \hat{Q}_h(\beta)\} = \frac{1}{n} \sum_{i=1}^n \{\frac{1}{\mathbb{E}} \sum_{j=1}^n i\} (x_i^T \beta - y_i)/h - \tau\} e_i w_i,
\]

so that \( \Delta^b(\beta) \) be written as

\[
\Delta^b(\beta) = \{G^b(\beta) - G^b(\beta^*)\} + \Delta(\beta),
\]

where \( \Delta(\beta) \) is defined in (60). By the triangle inequality,

\[
\sup_{\beta \in \beta^* + \Theta(r)} \|\Delta^b(\beta)\|_2 \leq \sup_{\beta \in \beta^* + \Theta(r)} \|G^b(\beta) - G^b(\beta^*)\|_2 + \sup_{\beta \in \beta^* + \Theta(r)} \|\Delta(\beta)\|_2.
\]

(79)

Let \( \mathcal{E}_3(t) \) denote the event that the bound (64) holds. It suffices to deal with the first term on the right-hand side of (79). Using a change of variable \( \delta = \Sigma^{1/2}(\beta - \beta^*) \in \mathbb{R}^p \) for \( \beta \in \beta^* + \Theta(r) \), we have \( y_i - x_i^T \beta = \epsilon_i - w_i^T \delta \) and

\[
\sup_{\beta \in \beta^* + \Theta(r)} \|G^b(\beta) - G^b(\beta^*)\|_2 \leq \sup_{\nu \in \mathbb{R}^p} \|G^b(\beta^* + \Sigma^{-1/2}\delta) - G^b(\beta^*)\|_2,
\]

(80)

where \( \Delta^b_0(\delta) = (1/n) \sum_{i=1}^n \epsilon_i w_i \{\tilde{K}(w_i^T \delta - \epsilon_i)/h) - \tilde{K}(-\epsilon_i/h)\} \) satisfies \( \Delta^b_0(0) = 0 \) and \( \mathbb{E}^*\{\Delta^b_0(\delta)\} = 0 \). Note that \( \nabla \Delta^b_0(\delta) = (1/n) \sum_{i=1}^n \epsilon_i K_i, \delta w_i w_i^T \), where \( K_i, \delta = K_i(w_i^T \delta - \epsilon_i) \).

For any \( \lambda \in \mathbb{R} \) and \( \mathbf{u}, \mathbf{v} \in S^{p-1} \), we have

\[
\mathbb{E}^* \exp\{\lambda n^{1/2} \mathbf{u}^T \Delta^b_0(\delta) \mathbf{v}\} = \prod_{i=1}^n \mathbb{E}^* \exp\{\lambda n^{1/2} \epsilon_i K_i, \delta w_i w_i^T \mathbf{u} \cdot w_i^T \mathbf{v}\}
\]

\[
\leq \prod_{i=1}^n \exp\{\lambda^2/2n K_i^2 \delta (w_i^T \mathbf{u} \cdot w_i^T \mathbf{v})^2\} = \exp\{\lambda^2/2n \sum_{i=1}^n K_i^2 (w_i^T \mathbf{u} \cdot w_i^T \mathbf{v})^2\}.
\]

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Note that $K_{i,\delta} \leq \kappa_u h^{-1}$, and by Hölder’s inequality,

\[
\frac{1}{n} \sum_{i=1}^{n} K_{i,\delta}(w_i^T u \cdot w_i^T v)^2 \leq \frac{\kappa_u}{h} \cdot \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,\delta}(w_i^T u)^4 \right\}^{1/2} \cdot \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,\delta}(w_i^T v)^4 \right\}^{1/2}.
\]

Define the function $\Lambda_{r,h}(\cdot, \cdot) : \mathbb{R}^p \times \mathbb{R}^p \mapsto [0, \infty)$

\[
\Lambda_{r,h}(u,v) = \frac{1}{n} \sum_{i=1}^{n} K_h(r w_i^T v - \varepsilon_i) (w_i^T u)^4 \quad \text{for } u, v \in \mathbb{S}^{p-1},
\]

and write $\|\Lambda_{r,h}\|_{\infty} = \sup_{u,v \in \mathbb{S}^{p-1}} \Lambda_{r,h}(u,v)$. With this notation, it follows that for any $\lambda \in \mathbb{R}$ and $u, v \in \mathbb{S}^{p-1}$,

\[
\sup_{\delta \in \mathbb{S}^{p}(r)} \mathbb{E}^* \exp\{ \lambda n^{1/2} \Lambda^{0\delta}(\delta) v \} \leq \exp \left\{ \frac{\lambda^2}{2} \kappa_u h^{-1} \|\Lambda_{r,h}\|_{\infty} \right\}.
\]

Thus, applying a conditional version of Theorem A.3 in Spokoiny (2013) yields

\[
\sup_{\delta \in \mathbb{S}^{p}(r)} \|\Lambda^{0\delta}(\delta)\|_2 \leq 6 \kappa_u^{1/2} \|\Lambda_{r,h}\|_{\infty}^{1/2} \cdot r \sqrt{\frac{4p + 2t}{nh}} \quad \text{(82)}
\]

with $\mathbb{P}^*$-probability at least $1 - e^{-t}$.

Next, we bound the data-dependent quantity $\|\Lambda_{r,h}\|_{\infty}$. For any $\epsilon_1, \epsilon_2 \in (0, 1)$, there exist $\epsilon_1$- and $\epsilon_2$-nets $\{ u_1, \ldots, u_{d_1} \}$ and $\{ v_1, \ldots, v_{d_2} \}$ of $\mathbb{S}^{p-1}$ with $d_1 \leq (1 + 2/\epsilon_1)^p$ and $d_2 \leq (1 + 2/\epsilon_2)^p$. Given $u, v \in \mathbb{S}^{p-1}$, there exist some $1 \leq j \leq d_1$ and $1 \leq k \leq d_2$ such that $\|u - u_j\|_2 \leq \epsilon_1$ and $\|v - v_k\|_2 \leq \epsilon_2$. At $(u, v)$, consider the decomposition

\[
\Lambda_{r,h}(u,v) = \Lambda_{r,h}(u,v) - \Lambda_{r,h}(u,v_k) + \Lambda_{r,h}(u,v_k).
\]

For $\Lambda_{r,h}(u,v) - \Lambda_{r,h}(u,v_k)$, the Lipschitz continuity of $K(\cdot)$ ensures that

\[
|\Lambda_{r,h}(u,v) - \Lambda_{r,h}(u,v_k)| \leq l_{K^r} \frac{n}{nh^2} \sum_{i=1}^{n} |w_i^T (v - v_k)(w_i^T u)|^4 \leq l_{K^r} \frac{\epsilon_2}{h^2} \cdot \max_{1 \leq i \leq n} \|w_i\|_2 \cdot \frac{1}{n} \sum_{i=1}^{n} (w_i^T u)^4. \quad \text{(83)}
\]

For $\Lambda_{r,h}(u,v_k)$, by the triangle inequality for the $L^4$-norm we have

\[
\Lambda_{r,h}^{1/4}(u,v_k) = \left\{ \frac{1}{n} \sum_{i=1}^{n} K_h(r w_i^T v_k - \varepsilon_i)(w_i^T u)^4 \right\}^{1/4} \leq \left\{ \frac{1}{n} \sum_{i=1}^{n} K_h(r w_i^T v_k - \varepsilon_i)(w_i^T u_j)^4 \right\}^{1/4} + \left\{ \frac{1}{n} \sum_{i=1}^{n} K_h(r w_i^T v_k - \varepsilon_i)(w_i^T u_j - u)^4 \right\}^{1/4} \leq \Lambda_{r,h}^{1/4}(u_j, v_k) + \epsilon_1 \cdot \sup_{u,v \in \mathbb{S}^{p-1}} \Lambda_{r,h}^{1/4}(u, v_k),
\]

which in turn implies

\[
\sup_{u,v \in \mathbb{S}^{p-1}} \Lambda_{r,h}(u,v_k) \leq (1 - \epsilon_1)^{-4} \max_{1 \leq j \leq d_1} \Lambda_{r,h}(u_j, v_k). \quad \text{(84)}
\]
In view of (83) and (84), it suffices to bound $\max_{1 \leq i \leq n} \|w_i\|_2$, $\sup_{u \in \mathbb{S}^{p-1}}(1/n) \sum_{i=1}^{n}(w_i^T u)^4$ and $\max_{(j,k) \in [d_1] \times [d_2]} \Lambda_{r,h}(u_j, v_k)$. By (76), the bound
\[
\max_{1 \leq i \leq n} \|w_i\|_2 \lesssim (p + \log n + t)^{1/2} \text{ holds on event } \mathcal{E}_{\text{max}}(t).
\]
For the supremum $\sup_{u \in \mathbb{S}^{p-1}}(1/n) \sum_{i=1}^{n}(w_i^T u)^4$, similarly to (84) it can be shown that
\[
\sup_{u \in \mathbb{S}^{p-1}} \frac{1}{n} \sum_{i=1}^{n}(w_i^T u)^4 \leq (1 - \epsilon_1)^{-4} \max_{1 \leq j \leq d_1} \frac{1}{n} \sum_{i=1}^{n}(w_i^T u_j)^4.
\]

Fix $j$ and $k$, Condition 3.5 implies
\[
\mathbb{E}_e\{ (w_j^T u_j)^4 / (36v_1^4) \}^{1/2} = \mathbb{E}_e\{ (w_j^T u_j)^2 / (6v_1^2) \} = 1 + \int_0^\infty e^u \mathbb{P}\{|w_j^T u_j| \geq v_1(6u)^{1/2}\} du \leq 1 + 2 \int_0^\infty e^{u-3u} du = 1 + 2 = 2.
\]
Therefore, $\| (w_j^T u_j)^4 \|_{\psi_{1/2}} \leq 36v_1^4$, where $\| \cdot \|_{\psi_r}$ denotes the $\psi_r$-norm $(r > 0)$; see Definition 2.1 in Adamczak et al. (2011). Since $0 \leq K_h(rw_j^T v_k - \varepsilon_i) \leq \kappa_u h^{-1}$, it is easy to see that $\|K_h(rw_j^T v_k - \varepsilon_i)(w_i^T u_j)^4\|_{\psi_{1/2}} \leq 36\kappa_u v_1^4 h^{-1}$. Moreover, note that $\mathbb{E}(w_i^T u_j)^4 \leq m_4$ and
\[
\mathbb{E}\{K_h(rw_j^T v_k - \varepsilon_i)(w_i^T u_j)^4\} = \mathbb{E}\{ \mathbb{E}\{K_h(rw_j^T v_k - \varepsilon_i)|x_i\}(w_i^T u_j)^4\} \leq \bar{f} m_4.
\]
Hence, for any $u, v \geq 3$, applying inequality (3.6) (and those above it) with $s = 1/2$ in Adamczak et al. (2011) and the union bound, we obtain that
\[
\mathbb{P}\left\{ \max_{1 \leq j \leq d_1} \frac{1}{n} \sum_{i=1}^{n}(w_i^T u_j)^4 \geq m_4 + C_1 v_1^4 \left( \frac{\sqrt{u}}{n} + \frac{u^2}{n^2} \right) \right\} \leq d_1 e^{-u}
\]
and
\[
\mathbb{P}\left\{ \max_{(j,k) \in [d_1] \times [d_2]} A(u_j, v_k) \geq \bar{f} m_4 + C_1 \kappa_u v_1^4 \left( \sqrt{\frac{v}{nh^2}} + \frac{v^2}{nh} \right) \right\} \leq d_1 d_2 e^{-v}.
\]
Taking $\epsilon_1 = 1 - 2^{-1/4}$, $\epsilon_2 = n^{-2}$, $u = p \log(1 + 2/\epsilon_1) + t$ and $v = p \log(1 + 2/\epsilon_1)(1 + 2/\epsilon_2) + t$ in the above bounds, it follows that with probability at least $1 - 2e^{-t}$,
\[
\max_{1 \leq j \leq d_1} \frac{1}{n} \sum_{i=1}^{n}(w_i^T u_j)^4 \leq m_4 + C_2 v_1^4 \left( \sqrt{\frac{p + t}{n}} + \frac{(p + t)^2}{n^2} \right)
\]
and
\[
\max_{(j,k) \in [d_1] \times [d_2]} A(u_j, v_k) \leq \bar{f} m_4 + C_3 \kappa_u v_1^4 \left( \sqrt{\frac{p \log n + t}{nh^2}} + \frac{(p \log n + t)^2}{nh^2} \right).
\]
Denote by $\mathcal{E}_4(t)$ the event that the above two bounds hold. For any $r \leq h$, conditioned on $\mathcal{E}_{\max}(t) \cap \mathcal{E}_4(t)$ with $t = \log n$, we have

$$\|A_{r,h}\|_\infty \leq 2 \max_{(j,k) \in [d_1] \times [d_2]} \Lambda(u_j, v_k) + \frac{2l_K r}{n^2} \cdot \max_{1 \leq i \leq n} \|w_i\|_2 \cdot \sup_{1 \leq j \leq d_1} \frac{1}{n} \sum_{i=1}^{n} (w_i^T u_j)^4$$

$$\leq 2 fm_4 + C_4 h^{-1} \left\{ \sqrt{\frac{p \log n}{n}} + \frac{(p \log n)^2}{n} \right\}. \tag{85}$$

With the above preparations, we are ready to prove the Bahadur representation for the bootstrap estimate. Let $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t) = \mathcal{E}_{\max}(t) \cap \mathcal{E}_{\text{smo}}(t) \cap \mathcal{E}_{\text{ls}}(t)$ be the events defined in the proof of Lemmas 3 and 4. In the rest of the proof, we take $t = \log n$ and set the bandwidth $h \asymp (q/n)^{2/5}$ with $q = p + \log n$. Then, conditioned on $\mathcal{E}_1(t) \cap \mathcal{E}_2(t)$, $\|\hat{\beta}_h - \beta^*\|_\Sigma \leq r_{\text{est}} \asymp \sqrt{q/n}$, and $\|\hat{\beta}_h - \beta^*\|_\Sigma \leq r_{\text{est}}^3 \asymp \sqrt{q/n}$ with $\mathbb{P}$-probability at least $1 - 2n^{-1}$. Conditioned further on $\mathcal{E}_3(t) \cap \mathcal{E}_4(t)$,

$$\|D(\hat{\beta}_h - \beta^*) + \nabla \hat{Q}_h(\beta^*)\|_\Omega = \|\Delta(\hat{\beta}_h)\|_2 \leq \sup_{\beta \in \beta^* + \Theta(r_{\text{est}})} \|\Delta(\beta)\|_2 \lesssim \left( \frac{q}{n} \right)^{4/5},$$

and with $\mathbb{P}$-probability at least $1 - 3n^{-1}$,

$$\|D(\hat{\beta}_h - \beta^*) + \nabla \hat{Q}_h(\beta^*)\|_\Omega = \|\Delta^\flat(\hat{\beta}_h)\|_2$$

$$\leq \sup_{\beta \in \beta^* + \Theta(r_{\text{est}}^3)} \|\Delta^\flat(\beta)\|_2 \lesssim \left( \frac{q}{n} \right)^{4/5} \sqrt{\left( \frac{q}{n} \right)^{3/5} \left( \frac{p \log n}{n} \right)^{1/4} \left( \frac{q}{n} \right)^{3/5} \frac{p \log n}{n^{1/2}}}.$$ 

Together, the above two bounds proves the claimed result. \qed

**Appendix C. Theoretical Properties of One-step Conquer**

In this section, we provide theoretical properties of the one-step conquer estimator $\hat{\beta}$, defined in Section 3.2. The key message is that, when higher-order kernels are used (and if the conditional density $f_{\varepsilon|x}(\cdot)$ has enough derivatives), the asymptotic normality of the one-step estimator holds under weaker growth conditions on $p$. For example, the scaling condition $p = o(n^{3/8})$ that is required for the conquer estimator can be reduced to roughly $p = o(n^{7/16})$ for the one-step conquer estimator using a kernel of order 4.

**Condition 1.** Let $G(\cdot)$ be a symmetric kernel of order $\nu > 2$, that is, $\int_{-\infty}^{\infty} u^k G(u) \, du = 0$ for $k = 1, \ldots, \nu - 1$ and $\int_{-\infty}^{\infty} u^\nu G(u) \, du \neq 0$. Moreover, $\kappa_{G,k} = \int_{-\infty}^{\infty} |u^k G(u)| \, du < \infty$ for $1 \leq k \leq \nu$, $G$ is uniformly bounded with $\kappa_{G,u} = \sup_{u \in \mathbb{R}} |G(u)| < \infty$ and is $l_G$-Lipschitz continuous for some $l_G > 0$.

The use of a higher-order kernel does not necessarily reduce bias unless the conditional density $f_{\varepsilon|x}(\cdot)$ is sufficiently smooth. Therefore, we further impose the following smoothness conditions on $f_{\varepsilon|x}(\cdot)$.

**Condition 2.** Let $\nu \geq 4$ be the integer in Condition 1. The conditional density $f_{\varepsilon|x}(\cdot)$ is $(\nu - 1)$-times differentiable, and satisfies $|f_{\varepsilon|x}^{(\nu-2)}(u) - f_{\varepsilon|x}^{(\nu-2)}(0)| \leq l_{\nu-2}|u|$ for all $u \in \mathbb{R}$.
almost surely (over the random vector $x$), where $l_{
u - 2} > 0$ is a constant. Also, there exists some constant $C_G > 0$ such that $\int_{-\infty}^{\infty} |u^{\nu - 1} G(u)| \cdot \sup_{|t| \leq |u|} |f_{\xi|\mathbf{x}}^{(\nu - 1)}(t) - f_{\mathbf{x}}^{(\nu - 1)}(0)| \, du \leq C_G$ almost surely.

Notably, we have

$$\nabla Q^G(\beta) = E\{\nabla Q((x, \beta) - y)/b) - \tau x\} \quad \text{and} \quad \nabla^2 Q^G(\beta) = E\{G_b(y - (x, \beta))xx^T\}, \quad (86)$$

representing the population score and Hessian of $Q^G(\cdot) = E\hat{Q}^G(\cdot)$. As $b \to 0$, we expect $\nabla Q^G(\beta^*)$ and $\nabla^2 Q^G(\beta^*)$ to converge to $0$ (zero vector in $\mathbb{R}^p$) and $D = E\{f_{\xi|\mathbf{x}}(0)xx^T\}$, respectively. Recall that $\Omega = \Sigma^{-1}$, and $\|u\|_\Omega = \|\Sigma^{-1/2}u\|_2$ for $u \in \mathbb{R}^p$. For any symmetric matrix $A \in \mathbb{R}^{p \times p}$, with slight abuse of notation we use $\|\cdot\|_\Omega$ to denote the relative operator norm, defined as $\|A\|_\Omega = \|\Sigma^{-1/2}A\Sigma^{-1/2}\|_2$.

The following proposition validates this claim by providing explicit error bounds.

**Proposition 2.** Let $b \in (0, 1)$ be a bandwidth. Under Conditions 1 and 2, we have

$$\|
abla Q^G(\beta^*)\|_\Omega \leq l_{\nu - 2}bG, \nu^b/v! \quad \text{and} \quad \|
abla^2 Q^G(\beta^*) - D_0\|_\Omega \leq C_G b^{\nu - 1}/(\nu - 1)!,$$

where $D_0 = \Sigma^{-1/2}D\Sigma^{-1/2} = E\{f_{\xi|\mathbf{x}}(0)ww^T\}$ with $w = \Sigma^{-1/2}x$.

Proposition 3 shows that when a higher-order kernel is used, the bias is significantly reduced in the sense that $\|
abla Q^G(\beta^*)\|_2 = O(b^\nu)$ and $\|
abla^2 Q^G(\beta^*) - D\|_2 = O(b^{\nu - 1})$, where $\nu \geq 4$ is an even integer. Notably, even if the kernel $G$ has negative parts, the population Hessian $\nabla^2 Q^G(\beta^*)$ preserves the positive definiteness of $D$ as long as the bandwidth $b$ is sufficiently small.

To construct the one-step conquer estimator, two key quantities are the sample Hessian $\nabla^2 \hat{Q}^G(\cdot)$ and sample gradient $\nabla \hat{Q}^G(\cdot)$, both evaluated at $\bar{\beta}$, a consistent initial estimate. In the next two propositions, we establish uniform convergence results of the Hessian and gradient of the empirical smoothed loss to their population counterparts. As a direct consequence, $\nabla^2 \hat{Q}^G(\bar{\beta})$ is positive definite with high probability, provided that $\bar{\beta}$ is consistent (i.e., in a local vicinity of $\beta^*$). To be more specific, for $r > 0$, we define the local neighborhood

$$\Theta^*(r) = \{\beta \in \mathbb{R}^p : \|\beta - \beta^*\|_\Omega \leq r\}. \quad (87)$$

**Proposition 3.** Conditions 1, 2 and 3.5 ensure that, with probability at least $1 - e^{-t}$,

$$\sup_{\beta \in \Theta^*(r)} \|
abla^2 \hat{Q}^G(\beta) - \nabla^2 Q^G(\beta)\|_\Omega \lesssim \sqrt{\frac{p\log n + t}{nb}} + \frac{p\log n + t}{nb} + \frac{(p + t)^{1/2}}{nb^2}$$

as long as $n \gtrsim p + t$.

**Proposition 4.** Conditions 1, 2 and 3.5 ensure that, with probability at least $1 - e^{-t}$,

$$\sup_{\beta \in \Theta^*(r)} \|
abla \hat{Q}^G(\beta) - \nabla \hat{Q}^G(\beta^*) - D(\beta - \beta^*)\|_\Omega \lesssim r \left(\sqrt{\frac{p + t}{nb}} + r + b^{\nu - 1}\right)$$

as long as $\sqrt{(p + t)/n} \lesssim b$. 

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With the above preparations, we are ready to present the Bahadur representation for the one-step conquer estimator $\hat{\beta}$.

**Theorem 6.** Assume Conditions 3.1, 3.2 and 3.5 in the main text and Conditions 1 and 2 hold. Let the bandwidths satisfy $n \geq p(\log n)^2 + t$, $\sqrt{(p+t)/n} \leq h \leq \{(p+t)/n\}^{1/4}$ and $\sqrt{(p+t)/n} \leq b \leq \{(p+t)/n\}^{1/(2\nu)}$. Then, the one-step conquer estimator $\hat{\beta}$ satisfies the bound

$$
\left\| \mathbf{D}(\hat{\beta} - \beta^*) - \frac{1}{n} \sum_{i=1}^{n} \{ t - \bar{G}(-\varepsilon_i/b) \} x_i \right\| \leq \Omega \left\{ \sqrt{(p \log n + t)/(nb)} + \frac{b^{\nu-1}}{n} \right\} \sqrt{\frac{p + t}{n}}
$$

(89)

with probability at least $1 - 5e^{-t}$, where $\bar{G}(u) = \int_{-\infty}^{u} G(v) \, dv$.

Theorem 6 shows that using a higher-order kernel ($\nu \geq 4$) allows one to choose larger bandwidth, thereby reducing the “variance” and the total Bahadur linearization error. Similarly to Theorem 3 in the main text, the following asymptotic normal approximation result for linear projections of one-step conquer is a direct consequence of Theorem 6.

**Theorem 7.** Assume Conditions 3.1, 3.2 and 3.5 in the main text and Conditions 1 and 2 hold. Let the bandwidths satisfy $(q/n)^{1/2} \leq h \leq (q/n)^{1/4}$ and $(q/n)^{1/2} \leq b \leq (q/n)^{1/(2\nu)}$, where $q := p + \log n$. Then,

$$
\sup_{x \in \mathbb{R}, a \in \mathbb{R}^p} \left| \mathbb{P}(n^{1/2}(a, \hat{\beta} - \beta^*) \leq \sigma x) - \Phi(x) \right| \leq \sqrt{\frac{(p + \log n) p \log n}{nb}} + n^{1/2} b^\nu,
$$

(90)

where $\sigma^2 = \tau(1 - \tau)\|\mathbf{D}^{-1} a\|_F^2$. In particular, with a choice of bandwidth $b \asymp (q/n)^{2/(2\nu+1)}$,

$$
\sup_{x \in \mathbb{R}, a \in \mathbb{R}^p} \left| \mathbb{P}(n^{1/2}(a, \hat{\beta} - \beta^*) \leq \sigma x) - \Phi(x) \right| \to 0
$$

as $n, p \to \infty$ under the scaling $p^{4\nu/(2\nu-1)}(\log n)^{(2\nu+1)/(2\nu-1)} = o(n)$.

Let $G(\cdot)$ be a kernel of order $\nu = 4$. In view of Theorem 7, we take $h \asymp \{(p + \log n)/n\}^{2/5}$ as in the main text and $b = \{(p + \log n)/n\}^{2/9}$, thereby obtaining that $n^{1/2}(a, \hat{\beta} - \beta)$, for an arbitrary $a \in \mathbb{R}^p$, is asymptotically normally distributed as long as $p(\log n)^{9/16} = o(n^{7/16})$ as $n \to \infty$.

**C.1 Proof of Proposition 2**

We start from the gradient $\Sigma^{-1/2} \nabla Q_{\varepsilon}^{\hat{G}}(\beta^*) = \mathbb{E}\{\bar{G}(-\varepsilon/b) - \tau\} w$ with $w = \Sigma^{-1/2} x$. Let $\mathbb{E}_x$ be the conditional expectation given $x$. By integration by parts,

$$
\mathbb{E}_x \bar{G}(-\varepsilon/b) = \int_{-\infty}^{\infty} \bar{G}(-t/b) \, dF_{\varepsilon|\tilde{x}}(t) = \int_{-\infty}^{\infty} G(u) F_{\varepsilon|\tilde{x}}(-bu) \, du.
$$

(91)
Applying a Taylor expansion with integral remainder on \( F_{\varepsilon|x}(bu) \) yields

\[
F_{\varepsilon|x}(bu) = F_{\varepsilon|x}(0) + \sum_{\ell=1}^{\nu-1} f_{\varepsilon|x}^{(\ell)}(0) \left( \frac{bu}{\ell!} \right)^\ell \left( \frac{bu}{(\ell+1)!} \right) + \frac{bu}{(\nu-2)!} \int_0^1 (1-w)^{\nu-2} \left\{ F_{\varepsilon|x}^{(\nu-1)}(bu) - F_{\varepsilon|x}^{(\nu-1)}(0) \right\} dw
\]

where

\[
\Gamma(2) = \int_0^\infty e^{-t} t^{(\nu-2)} dt = \frac{1}{\nu-2}.
\]

Recall that \( G \) is a kernel of order \( \nu \geq 4 \) (an even integer) and \( \kappa_{G,\nu} = \int_\infty |u^\nu G(u)| \, du < \infty \).

Substituting the above expansion into (91), we obtain

\[
E_x G(-\varepsilon/b) = \frac{b^{\nu-1}}{(\nu-2)!} \int_{-\infty}^\infty \int_0^1 u^{\nu-1} G(u)(1-w)^{\nu-2} \left\{ f_{\varepsilon|x}^{(\nu-2)}(bu) - f_{\varepsilon|x}^{(\nu-2)}(0) \right\} dw \, du.
\]

Furthermore, by the Lipschitz continuity of \( f_{\varepsilon|x}^{(\nu-2)}(\cdot) \) around 0,

\[
\| G_{\varepsilon/b} - \tau \| \leq \frac{b^{\nu-1}}{(\nu-2)!} \int_{-\infty}^\infty \int_0^1 u^{\nu-1} G(u)(1-w)^{\nu-2} w \, dw \, du = B(2,\nu-1) l_{\nu-2} \kappa_{G,\nu} b^\nu / (\nu-2)!,
\]

where \( B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt \) denotes the beta function. In particular, \( B(2,\nu-1) = \Gamma(2)\Gamma(\nu-1) / \Gamma(\nu + 1) = (\nu-1)! / \nu! \).

Putting together the pieces yields

\[
\| \nabla Q^G_b(\beta^*) \| \Omega = \sup_{u \in S^{p-1}} E_x \{ G_{\varepsilon/b} - \tau \} w^T u \leq b^{\nu-2} \kappa_{G,\nu} b^\nu / \nu!.
\]

Turning to the Hessian, note that

\[
\| \nabla^2 Q^G_b(\beta^*) - D_0 \| \Omega = \left\| E \int_{-\infty}^\infty G(u) \left\{ f_{\varepsilon|x}(bu) - f_{\varepsilon|x}(0) \right\} du \right\|.
\]

Applying a similar Taylor expansion as above, we have

\[
f_{\varepsilon|x}(t) = f_{\varepsilon|x}(0) + \sum_{\ell=1}^{\nu-1} f_{\varepsilon|x}^{(\ell)}(0) \frac{t^\ell}{\ell!} + \frac{t^{\nu-1}}{(\nu-2)!} \int_0^1 (1-w)^{\nu-2} \left\{ f_{\varepsilon|x}^{(\nu-1)}(tw) - f_{\varepsilon|x}^{(\nu-1)}(0) \right\} dw.
\]

Under Conditions 1 and 2, it follows that

\[
\| \nabla^2 Q^G_b(\beta^*) - D_0 \| \Omega \leq \frac{b^{\nu-1}}{(\nu-2)!} \sup_{u, \delta \in S^{p-1}} E \int_{-\infty}^\infty \int_0^1 u^{\nu-1} G(u)(1-w)^{\nu-2} \left\{ f_{\varepsilon|x}^{(\nu-1)}(bu) - f_{\varepsilon|x}^{(\nu-1)}(0) \right\} dw \, du \, \langle w, \delta \rangle
\]

\[
\leq \frac{b^{\nu-1}}{(\nu-1)!} \sup_{u, \delta \in S^{p-1}} E \int_{-\infty}^\infty \int_0^1 u^{\nu-1} G(u) \sup_{|\delta| \leq \|u\|} |f_{\varepsilon|x}^{(\nu-1)}(t) - f_{\varepsilon|x}^{(\nu-1)}(0)| \, du \cdot \langle w, \delta \rangle \|w\| \|
\]

\[
\leq C_G b^{\nu-1} \frac{(\nu-1)!}{(\nu-1)!} \sup_{u \in S^{p-1}} E\|w\|^2 = \frac{C_G}{(\nu-1)!} b^{\nu-1}.
\]

This completes the proof.
C.2 Proof of Proposition 3

Consider the change of variable \( \delta = \Sigma^{1/2}(\beta - \beta^*) \), so that \( \beta \in \Theta^*(r) \) is equivalent to \( \delta \in \mathbb{B}^p(r) \). Write \( w_i = \Sigma^{-1/2}x_i \in \mathbb{R}^p \), which are isotropic random vectors, and define

\[
H_n(\delta) = \Sigma^{-1/2}\nabla^2 \tilde{Q}_b^G(\beta) \Sigma^{-1/2} = \frac{1}{n} \sum_{i=1}^{n} G_b(\varepsilon_i - w_i^T \delta)w_i w_i^T, \quad H(\delta) = \mathbb{E}\{H_n(\delta)\}. \tag{93}
\]

For any \( \epsilon_1 \in (0, r) \), there exists an \( \epsilon \)-net \( \mathcal{N}_1 := \{\delta_1, \ldots, \delta_{d_1}\} \) with \( d_1 \leq (1+2r/\epsilon_1)^p \) satisfying that, for every \( \delta \in \mathbb{B}^p(r) \), there exists some \( 1 \leq j \leq d_1 \) such that \( \|\delta - \delta_j\|_2 \leq \epsilon_1 \). Hence,

\[
\|H_n(\delta) - H(\delta)\|_2 \\
\leq \|H_n(\delta) - H_n(\delta_j)\|_2 + \|H_n(\delta_j) - H(\delta_j)\|_2 + \|H(\delta_j) - H(\delta)\|_2 \\
=: I_1(\delta) + I_2(\delta_j) + I_3(\delta).
\]

For \( I_1(\delta) \), note that \( G_b(u) = (1/b)G(u/b) \) is Lipschitz continuous, i.e. \( |G_b(u) - G_b(v)| \leq lGb^{-2}|u - v| \) for all \( u, v \in \mathbb{R} \). It follows that

\[
I_1(\delta) \leq \sup_{u, v \in \mathcal{S}^{p-1}} \frac{1}{n} \sum_{i=1}^{n} |G_b(\varepsilon_i - w_i^T \delta) - G_b(\varepsilon_i - w_i^T \delta_j)| \cdot |w_i^T u \cdot w_i^T v| \\
\leq lGb^{-2} \sup_{u, v \in \mathcal{S}^{p-1}} \frac{1}{n} \sum_{i=1}^{n} |w_i^T (\delta - \delta_j) \cdot w_i^T u \cdot w_i^T v| \\
\leq lGb^{-2}\epsilon_1 \sup_{u \in \mathcal{S}^{p-1}} \frac{1}{n} \sum_{i=1}^{n} |w_i^T u|^3. \tag{94}
\]

Next, we use the standard covering argument to bound \( M_{n,3} \). Given \( \epsilon_2 \in (0, 1) \), let \( \mathcal{N}_2 \) be an \( \epsilon_2 \)-net of the unit sphere \( \mathcal{S}^p \) with \( d_2 := |\mathcal{N}_2| \leq (1+2/\epsilon_2)^p \) such that for every \( u \in \mathcal{S}^p \), there exists some \( v \in \mathcal{N}_2 \) satisfying \( \|u - v\|_2 \leq \epsilon_2 \). Define the (standardized) design matrix \( W_n = n^{-1/3}(w_1, \ldots, w_n)^T \in \mathbb{R}^{n \times p} \), so that \( M_{n,3} = \sup_{u \in \mathcal{S}^p} \|W_n u\|_3 \). By the triangle inequality,

\[
\|W_n u\|_3 \leq \|W_n v\|_3 + \|W_n(u - v)\|_3 \\
= \|W_n \delta\|_3 + \left( \frac{1}{n} \sum_{i=1}^{n} |w_i^T (u - v)|^3 \right)^{1/3} \leq \|W_n v\|_3 + \epsilon_2 \cdot M_{n,3}^{1/3}.
\]

Taking the maximum over \( v \in \mathcal{N}_2 \), and then taking the supremum over \( u \in \mathcal{S}^p \), we arrive at

\[
M_{n,3} \leq (1 - \epsilon_2)^{-3} \cdot N_{n,3} := (1 - \epsilon_2)^{-3} \cdot \max_{v \in \mathcal{N}_2} \frac{1}{n} \sum_{i=1}^{n} |w_i^T v|^3. \tag{95}
\]

For every \( v \in \mathcal{N}_2 \), note that

\[
\mathbb{E}e(|w_i^T v|^3/(6^{3/2}v_i^3))^{2/3} = 1 + \int_0^\infty e^{u} \mathbb{P}\{|w_i^T v| \geq v_1(6u)^{1/2}\} du \leq 1 + 2 \int_0^\infty e^{-2u} du = 2,
\]

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implying \( \| |w_i^T v| \|^3_{\psi_{2/3}} \leq 6^{3/2} v_1^3 \). Hence, by inequality (3.6) in Adamczak et al. (2011) with \( s = 2/3 \), we obtain that for any \( z \geq 3 \),

\[
\frac{1}{n} \sum_{i=1}^{n} |w_i^T v|^3 \leq E|w_i^T v|^3 + C_1 \left( \sqrt{\frac{z}{n}} + \frac{z^{3/2}}{n} \right)
\]

with probability at least \( 1 - e^{-z} \), where \( C_1 > 0 \) is a constant depending only on \( v_1 \). Taking the union bound over all vectors \( v \) in \( N_2 \) yields that, with probability at least \( 1 - d_2 e^{-z} \geq 1 - e^{p \log(1+2/\epsilon_2) - z} \),

\[
N_{n,3} \leq m_3 + C_1 \left( \sqrt{\frac{z}{n}} + \frac{z^{3/2}}{n} \right)
\]

where \( m_3 = \sup_{u \in \mathbb{S}^{p-1}} E|w_i^TU|^3 \). Reorganizing the terms, we get

\[
N_{n,3} \leq m_3 + C_1 \left[ \frac{p \log(1+2/\epsilon_2) + \log 2 + t}{n} + \frac{p \log(1+2/\epsilon_2) + \log 2 + t^{3/2}}{n} \right]
\]

with probability at least \( 1 - e^{-t/2} \). Taking \( \epsilon_2 = 1/8 \) in (95) and (96) implies

\[
M_{n,3} \leq 1.5m_3 + 1.5C_1 \left\{ \sqrt{\frac{3p + 1 + t}{n}} + \frac{(3p + 1 + t)^{3/2}}{n} \right\}.
\]

Under the sample size scaling \( n \geq p + t \), plugging the above bound into (94) yields

\[
\sup_{\delta \in \mathbb{B}^p} I_1(\delta) \lesssim (p + t)^{1/2} b^{-2} \epsilon_1
\]

with probability at least \( 1 - e^{-t/2} \). For \( I_3(\delta) \), it can be similarly obtained that

\[
I_3(\delta) \leq l_G b^{-2} \sup_{u, w \in \mathbb{S}^{p-1}} E|w_i^T (\delta - \delta_j) \cdot w_i^T u| \leq l_G m_3 b^{-2} \epsilon_1
\]

uniformly over all \( \delta \in \mathbb{B}^p \).

Turning to \( I_2(\delta_j) \), note that \( H_n(\delta_j) - H(\delta_j) = (1/n) \sum_{i=1}^{n} (1 - E) \phi_{ij} w_i w_i^T \), where \( \phi_{ij} = G_0(\epsilon_{ij} - w_i^T \delta_j) \) satisfy \( |\phi_{ij}| \leq \kappa_{G, u} b^{-1} \) and

\[
E(\phi_{ij}^2 | x_i) = \frac{1}{b^2} \int_{-\infty}^{\infty} G^2 \left( \frac{w_i - t}{b} \right) f_{\epsilon_{ij}}(t) \, dt = \frac{1}{b} \int_{-\infty}^{\infty} G^2(u) f_{\epsilon_{ij}}(w_i^T \delta - bu) \, du \leq \frac{\tilde{f}_{G^2, 0}}{b}
\]

almost surely, where \( \kappa_{G^2, 0} := \int_{-\infty}^{\infty} G^2(u) \, du < \infty \). Given \( \epsilon_3 \in (0, 1/2) \), there exists an \( \epsilon_3 \)-net \( N_3 \) of the sphere \( \mathbb{S}^{p-1} \) with \( |N_3| \leq (1 + 2/\epsilon_3)^p \) such that \( \| H_n(\delta_j) - H(\delta_j) \|_2 \leq (1 - 2\epsilon_3)^{-1} \max_{u \in \mathcal{M}} |u^T \{ H_n(\delta_j) - H(\delta_j) \} u| \). Given \( u \in N_3 \) and \( k = 2, 3, \ldots \), we bound the higher order moments of \( \phi_{ij}(w_i^T u)^2 \) by

\[
E[\phi_{ij}(w_i^T u)^2]^k \leq \tilde{f}_{G^2, 0} b^{-1} \cdot (\kappa_{G, u} b^{-1})^k \cdot 2^k \cdot 2k \cdot \int_{0}^{\infty} \mathbb{P}(|w_i^T u| \geq v_1 u) u^{2k-1} \, du
\]

\[
\leq \tilde{f}_{G^2, 0} b^{-1} \cdot (\kappa_{G, u} b^{-1})^k \cdot 4k \cdot \int_{0}^{\infty} u^{2k-1} e^{-u^2/2} \, du
\]

\[
\leq \tilde{f}_{G^2, 0} b^{-1} \cdot (\kappa_{G, u} b^{-1})^k \cdot 2^{k+1} k!.
\]
we obtain that with probability at least 1
\[ 1 - \frac{2\nu^2_1}{1 - 2\epsilon_3} \]
the proof, we conclude that with probability at least 1
\[ 1 - \frac{2\nu^2_1}{1 - 2\epsilon_3} \]
Once again, taking the union bound over
\[ j \]
Define the stochastic process \( \Delta_b \nabla \) as long as
\[ \max_{1 \leq j \leq d_1} I_2(\delta_j) \lesssim \sqrt{\frac{p + v}{nb}} + \frac{p + v}{nb}. \]
with probability at least 1
\[ 1 - 2(1 + 2/\epsilon_3)^p e^{-u} = 1 - (1/2)e^{\log(4)+p\log(1+2/\epsilon_3)-u}. \]
Setting \( \epsilon_3 = 2/(e^3 - 1) \) and \( u = \log(4) + 3p + v \), it follows that with probability at least 1
\[ 1 - e^{-v/2}, \]
\[ I_2(\delta_j) \lesssim \sqrt{\frac{p + v}{nb}} + \frac{p + v}{nb}. \]
Once again, taking the union bound over \( j = 1, \ldots, d_1 \) and setting \( v = \log(1 + 2r/\epsilon_1) + t \), we obtain that with probability at least 1
\[ 1 - d_1 e^{-v} \geq 1 - e^{-t/2}, \]
\[ \max_{1 \leq j \leq d_1} I_2(\delta_j) \lesssim \sqrt{\frac{p \log(3r/\epsilon_1) + t}{nb}} + \frac{p \log(3r/\epsilon_1) + t}{nb}. \]
Finally, combining (97), (98) and (99), and taking \( \epsilon_1 = r/n \in (0, r) \) in the beginning of the proof, we conclude that with probability at least 1
\[ 1 - e^{-t}, \]
\[ \sup_{\beta \in \Theta^*(r)} \| \nabla^2 \hat{Q}^G_b(\beta) - \nabla^2 \hat{Q}^G_b(\beta) \|_\Omega \lesssim \sqrt{\frac{p \log n + t}{nb}} + \frac{p \log n + t}{nb} + \frac{(p + t)^{1/2}r}{nb^2} \]
as long as \( n \gtrsim p + t \). This completes the proof.

\[ \Box \]

C.3 Proof of Proposition 4

Define the stochastic process \( \Delta_b(\beta) = \Sigma^{-1/2}\{ \nabla \hat{Q}^G_b(\beta) - \nabla \hat{Q}^G_b(\beta^*) - D(\beta - \beta^*) \} \). By the triangle inequality,
\[ \sup_{\beta \in \Theta^*(r)} \| \Delta_b(\beta) \|_2 \leq \sup_{\beta \in \Theta^*(r)} \| \mathbb{E}\Delta_b(\beta) \|_2 + \sup_{\beta \in \Theta^*(r)} \| \Delta_b(\beta) - \mathbb{E}\Delta_b(\beta) \|_2 \]
Recall that \( D_0 = \Sigma^{-1/2} D \Sigma^{-1/2} = \mathbb{E}\{ f_{\epsilon|x}(0) w w^T \} \). For the first term on the right-hand side, using the mean value theorem for vector-valued functions yields
\[ \mathbb{E}\Delta_b(\beta) = \left\{ \Sigma^{-1/2} \int_0^1 \nabla^2 \hat{Q}^G_b((1 - s)\beta^* + s\beta) \Sigma^{-1/2} - D_0 \right\} \Sigma^{1/2}(\beta - \beta^*). \]
By a change of variable \( \delta = \Sigma^{1/2}(\beta - \beta^*), \)
\[ \nabla^2 \hat{Q}^G_b((1 - s)\beta^* + s\beta) = \mathbb{E} \int_{-\infty}^\infty G(u) f_{\epsilon|x}(sw^T \delta - bu) \, du \cdot xx^T. \]
For every $s \in [0, 1]$ and $u \in \mathbb{R}$, it ensures from $f_{c|x}(\cdot)$ being Lipschitz that
\[ |f_{c|x}(sw^\top \delta - bu) - f_{c|x}(-bu)| \leq l_0 s \cdot |w^\top \delta|. \]

Moreover, by the Taylor expansion (92),
\[ f_{c|x}(-bu) = f_{c|x}(0) + \sum_{\ell=1}^{\nu-1} f_{c|x}(0) (-bu)^\ell \ell! + (-bu)^{\nu-1} (1 - w)^{\nu-2} \{ f_{c|x}^{(\nu-1)} (-bw) - f_{c|x}^{(\nu-1)} (0) \} \, dw. \]

Consequently,
\[
\begin{align*}
\left\| \Sigma^{-1/2} \int_0^1 \nabla^2 Q_b^G ((1 - s) \beta^* + s \beta) \, ds \Sigma^{-1/2} - D_0 \right\|_2 & \\
& \leq \left\| \frac{b^{\nu-1}}{(\nu - 2)!} \int_{-\infty}^{\infty} \int_0^1 u^{\nu-1} G(u) (1 - w)^{\nu-2} \{ f_{c|x}^{(\nu-1)} (-bw) - f_{c|x}^{(\nu-1)} (0) \} \, dwu \cdot wu^\top \right\|_2 \\
& \quad + 0.5l_0 \cdot \| \Sigma w^\top \delta \cdot wu^\top \|_2 \\
& \leq \frac{C_G b^{\nu-1}}{(\nu - 1)!} \sup_{u \in \mathbb{S}^p-1} \mathbb{E}(w, u)^2 + \frac{l_0}{2} \sup_{u \in \mathbb{S}^p-1} \mathbb{E}(|w^\top \delta| (w, u)^2) \\
& \leq \left( \frac{C_G}{(\nu - 1)!} b^{\nu-1} + 0.5l_0 m_3 \| \delta \|_2 \right).
\end{align*}
\]

Taking the supremum over $\beta \in \Theta^*(r)$, or equivalently $\delta \in \mathcal{B}_r^p(\cdot)$, yields
\[
\sup_{\beta \in \Theta^*(r)} \| \mathbb{E} \Delta_b(\beta) \| \leq \left( \frac{C_G}{(\nu - 1)!} b^{\nu-1} + 0.5l_0 m_3 r \right) r \lesssim (b^{\nu-1} + r)r.
\]

For the stochastic term $\sup_{\beta \in \Theta^*(r)} \| \Delta_b(\beta) - \mathbb{E} \Delta_b(\beta) \|_2$, following the proof of Theorem 2, it can be similarly shown that with probability at least $1 - e^{-t}$,
\[
\sup_{\beta \in \Theta^*(r)} \| \Delta_b(\beta) - \mathbb{E} \Delta_b(\beta) \|_2 \lesssim r \sqrt{\frac{p + t}{nb}}
\]
as long as $\sqrt{(p + t)/n} \lesssim b$.

Combining the last two displays completes the proof of (88). \qed

### C.4 Proof of Theorem 6

**Step 1** (Consistency of the initial estimate). First, note that the consistency of the initial estimator $\mathbf{\beta}$—namely, $\mathbf{\beta}$ lies in a local neighborhood of $\beta^*$ with high probability, is a direct consequence of Theorem 1. Given a non-negative kernel $K(\cdot)$ and for any $t > 0$, the initial estimator $\mathbf{\beta}$ satisfies
\[
\| \mathbf{\beta} - \beta^* \|_\Sigma \leq r_n \times \sqrt{\frac{p + t}{n}} \quad (100)
\]
with probability at least $1 - 2e^{-t}$ as long as $(\frac{p + t}{n})^{1/2} \lesssim h \lesssim (\frac{p + t}{n})^{1/4}$. Let $E_{\text{init}}(t)$ be the event that (100) holds. Provided that the sample Hessian $\nabla^2 \hat{Q}^G_b(\beta)$ is invertible, we have

$$
\mathbf{D}(\hat{\beta} - \beta) = -\mathbf{D}\{\nabla^2 \hat{Q}^G_b(\beta)\}^{-1} \nabla \hat{Q}^G_b(\beta)
$$

or equivalently,

$$
\Sigma^{-1/2}\mathbf{D}(\hat{\beta} - \beta^s) = -\mathbf{D}_0\mathbf{D}_0^{-1} \cdot \Sigma^{-1/2}\{\nabla \hat{Q}^G_b(\beta) - \nabla \hat{Q}^G_b(\beta^s) - \mathbf{D}(\beta - \beta^s)\}
$$

$$+ (I_p - \mathbf{D}_0\mathbf{D}_0^{-1})\mathbf{D}_0 \cdot \Sigma^{1/2}(\beta - \beta^s) - \mathbf{D}_0\mathbf{D}_0^{-1} \cdot \Sigma^{-1/2}\nabla \hat{Q}^G_b(\beta^s),
$$

where $\mathbf{D}_0 = \mathbb{E}\{f_{x|x}(0)ww^T\} = \Sigma^{-1/2}\mathbf{D}\Sigma^{-1/2}$ and $\mathbf{D}_0 := \Sigma^{-1/2}\nabla^2 \hat{Q}^G_b(\beta)\Sigma^{-1/2}$. It follows that

$$
||\mathbf{D}(\hat{\beta} - \beta^s) + \nabla \hat{Q}^G_b(\beta^s)||_{\Omega}
\leq ||(I_p - \mathbf{D}_0\mathbf{D}_0^{-1})\mathbf{D}_0||_2 \cdot ||\beta - \beta^s||_{\Sigma} + ||I_p - \mathbf{D}_0\mathbf{D}_0^{-1}||_2 \cdot ||\nabla \hat{Q}^G_b(\beta^s)||_{\Omega}
+ ||\mathbf{D}_0\mathbf{D}_0^{-1}||_2 \cdot ||\nabla \hat{Q}^G_b(\beta) - \nabla \hat{Q}^G_b(\beta^s) - \mathbf{D}(\beta - \beta^s)||_{\Omega}.
$$

In view of (101), it remains to bound the following three quantities:

$$
||I_p - \mathbf{D}_0\mathbf{D}_0^{-1}||_2, \ ||\nabla \hat{Q}^G_b(\beta^s)||_{\Omega} \text{ and } ||\nabla \hat{Q}^G_b(\beta) - \nabla \hat{Q}^G_b(\beta^s) - \mathbf{D}(\beta - \beta^s)||_{\Omega}.
$$

**STEP 2** (Consistency of the sample Hessian $\nabla^2 \hat{Q}^G_b(\beta)$). Recall that $\tilde{D}_0 = \Sigma^{-1/2}\nabla^2 \hat{Q}^G_b(\beta)\Sigma^{-1/2}$ and $\mathbf{D}_0 = \Sigma^{-1/2}\mathbf{D}\Sigma^{-1/2}$. By the triangle inequality,

$$
||\tilde{D}_0 - \mathbf{D}_0||_2 \leq ||\nabla^2 \hat{Q}^G_b(\beta) - \nabla^2 \hat{Q}^G_b(\beta^s)||_{\Omega} + ||\nabla^2 \hat{Q}^G_b(\beta^s) - \mathbf{D}||_{\Omega}.
$$

Let the bandwidth $b$ satisfy $\max\{\frac{b\log n + t}{n}, n^{-1/2}\} \lesssim b \lesssim 1$. Conditioned on $E_{\text{init}}(t)$, applying Propositions 2 and 3 with $r = r_n$ yields that, with probability $1 - e^{-t},$

$$
||\tilde{D}_0 - \mathbf{D}_0||_2 \leq \delta_n \times \sqrt{\frac{p\log n + t}{nb}} + b^{-1}.
$$

Under Condition 3.2, $0 < L \leq \lambda_{\text{min}}(\mathbf{D}_0) \leq \lambda_{\text{max}}(\mathbf{D}_0) \leq \tilde{f}$, so that $||\mathbf{D}_0^{-1}||_2 \leq \tilde{f}^{-1}$. For sufficiently large $n$ and small $b$, this implies $||\tilde{D}_0\mathbf{D}_0^{-1} - I_p||_2 \leq \tilde{f}^{-1}\delta_n < 1$, and hence

$$
||\mathbf{D}_0\mathbf{D}_0^{-1} - I_p||_2 \leq \frac{\delta_n}{\tilde{f} - \delta_n}, \quad ||\mathbf{D}_0\mathbf{D}_0^{-1}||_2 \leq \frac{\tilde{f}}{\tilde{f} - \delta_n}.
$$

**STEP 3** (Controlling the scores). For $||\nabla \hat{Q}^G_b(\beta^s)||_{\Omega}$, it follows from Lemma 1 and Proposition 2 that with probability at least $1 - e^{-t},$

$$
||\nabla \hat{Q}^G_b(\beta^s)||_{\Omega} \lesssim \sqrt{\frac{p + t}{n}} + b^\nu.
$$
Turning to $\|\nabla \hat{Q}_b^G(\beta) - \nabla \hat{Q}_b^G(\beta^*) - D(\beta - \beta^*)\|_\Omega$, applying the concentration bound (100) and Proposition 4 we obtain that, with probability at least $1 - e^{-t}$ conditioned on $\mathcal{E}_{\text{init}}(t)$,

$$\|\nabla \hat{Q}_b^G(\beta) - \nabla \hat{Q}_b^G(\beta^*) - D(\beta - \beta^*)\|_\Omega \lesssim \frac{p+t}{nb^{1/2}} + b^{\nu-1}\sqrt{\frac{p+t}{n}}$$

(104)

as long as $(\frac{p+t}{n})^{1/2} \lesssim b \lesssim 1$.

Finally, combining the bounds (100)–(104), we conclude that with probability at least $1 - 5e^{-t}$,

$$\|D(\hat{\beta} - \beta^*) + \nabla \hat{Q}_b^G(\beta^*)\|_\Omega \lesssim \left(\sqrt{\frac{p\log n + t}{nb}} + b^{\nu-1}\right)\left(\sqrt{\frac{p+t}{n}} + b^{\nu}\right),$$

provided that $\max\left\{\frac{p\log n + t}{n}, (\frac{p+t}{n})^{1/2}\right\} \lesssim b \lesssim 1$. Under the sample size scaling $n \gtrsim p(\log n)^2 + t$ and bandwidth constraint $b \lesssim (\frac{p+t}{n})^{1/(2\nu)}$, this leads to the claimed bound (89). \qed

**Appendix D. Additional Simulation Results**

In this section, we present additional results for the numerical studies as described in Section 5 in Figures 9–13.
References


Figure 9: Results under models (29)–(31) in Section 5 with \( \tau = \{0.1, 0.3, 0.5, 0.7\} \) and \( t_2 \) noise, averaged over 100 simulations. This figure extends the last row of Figure 3 to other quantile levels.
Figure 10: Elapsed time of standard QR, Horowitz's smoothing and conquer under \( t_2 \)-noise setting when \( \tau \in \{0.1, 0.3, 0.5, 0.7\} \). This figure extends the last row of Figure 4 to other quantile levels.
Figure 11: Empirical coverage, confidence interval width, and elapsed time of six methods with $N(0, 4)$ errors under $\tau = 0.9$. Other details are as in Figure 6.
Figure 12: Empirical coverage, confidence interval width, and elapsed time of six methods with $t_2$ errors under $\tau = 0.5$. Other details are as in Figure 6.
Figure 13: Empirical coverage, confidence interval width, and elapsed time of six methods with $\mathcal{N}(0, 4)$ errors under $\tau = 0.5$. Other details are as in Figure 6.