PRELIMINARY DRAFT

An inhomogeneus reverse ergodic theorem and application to a new uniqueness result for reflecting diffusions

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January, 25 2021

1 Introduction

Let E be a compact metric space, and let $Q(x, dy)$ be a sub-probability transition function on E, that is $Q(x, dy)$ satisfies all conditions for a probability transition function except that $Q(x, E) \leq 1$. In the proof of their Theorem 5.4, Bass and Pardoux (1987) show that, if Q satisfies the conditions of the Krein-Rutman theorem (Theorems 6.1 and 6.3 of Kreĭn and Rutman (1950)), then, for any pair of continuous functions f, g, $g > 0$ [1], and any sequence of probability measures on E, $\{\nu_k\}$,

$$\lim_{k\to\infty} \frac{\int Q^k f(x)\,\nu_k(dx)}{\int Q^k g(x)\,\nu_k(dx)} = C(f, g), \tag{1.1}$$

where the constant $C(f, g)$ is independent of the sequence $\{\nu_k\}$. In particular, with $g \equiv 1$,

$$\lim_{k\to\infty} \frac{\int Q^k f(x)\,\nu_k(dx)}{\int Q^k \mathbf{1}(x)\,\nu_k(dx)} = C(f).$$

(1.1) can be viewed as a *reverse ergodic theorem* for sub-probability transition functions. Note that, tipically, both the numerator and the denominator in (1.1) tend to zero.

The result of Bass and Pardoux (1987) is a key element in the proof of uniqueness of the reflecting Brownian motion in a cone, with radially constant direction of reflection, by

[1] The conditions of the Krein-Rutman theorem actually allow for $g \geq 0$ as long as g is not identically zero

Kwon and Williams (1991) and in the proof of uniqueness of reflecting Brownian motion in an orthant, with constant directions of reflection on each face, by Taylor and Williams (1993).

Our first goal here is to extend the Bass and Pardoux (1987) result to a sequence of compact metric spaces $E_0, E_1, E_2, \dots$ and a sequence of sub-transition functions $Q_1, Q_2, \dots$, with Q_l governing transitions from E_l to E_{l-1}, and give conditions under which

$$\lim_{k\to\infty} \frac{\int Q_k Q_{k-1} \cdots Q_1 f(x) \nu_k(dx)}{\int Q_k Q_{k-1} \cdots Q_1 g(x) \nu_k(dx)} = C(f,g) \tag{1.2}$$

where $C(f,g)$ is independent of $\{\nu_k\}$. We call (1.2) an *inhomogeneous reverse ergodic theorem* for sub-probability transition functions.

Note that, even in the case when $E_l = E$ for all l and Q_l converges, as l goes to infinity, to a sub-transition function Q on E, it is not in general possible to reconduct the limit in (1.2) to the Krein-Rutman theorem. In fact this would essentially require exchanging the limits

$$\lim_{l\to\infty} \lim_{k\to\infty} \frac{\int Q_{l+k} Q_{l+k-1} \cdots Q_{l+1} f(x) \nu_k(dx)}{\int Q_{l+k} Q_{l+k-1} \cdots Q_{l+1} g(x) \nu_k(dx)} = \lim_{k\to\infty} \lim_{l\to\infty} \frac{\int Q_{l+k} Q_{l+k-1} \cdots Q_{l+1} f(x) \nu_k(dx)}{\int Q_{l+k} Q_{l+k-1} \cdots Q_{l+1} g(x) \nu_k(dx)}.$$

Rather than trying to reinforce the conditions of the Krein-Rutman theorem, we provide new conditions under which (1.2) holds (Theorem 2.4). Our conditions are uniform lower bounds which have a clear probabilistic meaning and can be verified in many contexts.

In fact our second goal is to use our inhomogeneous reverse ergodic theorem to prove uniqueness for a reflecting diffusion process, with varying, oblique direction of reflection, in a curved domain with only one singular point, that, in a neighborhood of the singular point, looks like a cone. Although one expects that such a diffusion, for a short time after it leaves the singular point, is essentially a reflecting Brownian motion in the cone (we actually prove this in a rigorous sense: see Lemma 3.28), it is not clear how the uniqueness result we seek might follow from the Kwon and Williams (1991) result, essentially due to a limit exchange problem like the one mentioned above.

With our approach, we can prove uniqueness under very natural, geometric conditions (Conditions 3.1, 3.3 and 3.6): Besides mild regularity conditions, we only require that, at the singular point, where we have not a single normal direction and a single direction of reflection, but a cone of normal directions and a cone of directions of reflection, there is a normal vector that forms an angle of strictly less than $\pi/2$ with every direction of reflection, and a direction of reflection that forms an angle of strictly less than $\pi/2$ with every normal direction (Conditions 3.3 (*iii*) and (*iv*)).

Our argument follows the general outline of Kwon and Williams (1991), with two fundamental changes: We characterize a reflecting diffusion as a solution of a *constrained martingale problem* rather than of a submartingale problem, which simplifies all compactness arguments; We replace the Krein-Rutman theorem by our inhomogeneous reverse ergodic theorem.

In order to apply our inhomogeneous reverse ergodic theorem, first of all we note that we can apply the Markov selection results of Costantini and Kurtz (2019), so that we can reduce to proving uniqueness among strong Markov reflecting diffusions.

Next, we need to prove our uniform lower bounds: (i) an (ii) in Theorem 2.4. We obtain the bound (i) by means of some auxiliary functions that we construct by elaborating on the functions ψ_α and χ introduced in Kwon and Williams (1991) (or the corresponding functions introduced in Varadhan and Williams (1985), in the 2-dimensional case). In order to do this, we have to prove that, under our conditions, the parameter α in Kwon and Williams (1991) - which rules the probability that the reflecting Brownian motion in the cone hits the tip - satisfies $\alpha < 1$. This follows essentially by the fact that, under our conditions, the reflecting Brownian motion is a semimartingale (Theorem A1.9).

In order to verify the bound (ii), we use a coupling argument based on Lemma 5.3 of Costantini and Kurtz (2018) (Lemma 3.29), the fact that, for any reflecting diffusion, X, the rescaled process $2^{2n}\,X(2^{-4n}\cdot)$ converges to the reflecting Brownian motion in the cone (the above mentioned Lemma 3.28), and the support theorem of Kwon and Williams (1991).

Finally, let us mention that we are currently working on a general existence and uniqueness result for reflecting diffusions in a piecewise smooth domain in dimension 2, which exploits the results presented here.

The outline of the paper is the following: in Section 2 we prove our inhomogeneous reverse ergodic theorem, while in Section 3 we prove our uniqueness result for reflecting diffusions. Section 3 is divided into several subsections: in Subsection 3, we formulate the problem and the assumptions, and we prove some preliminary results; in Subsection 3.2 we prove existence of a strong Markov, reflecting diffusion; in Subsection 3.3 we outline the proof of uniqueness; in Subsections 3.4 and 3.5 we prove the required bounds. Finally in Appendix A1 we summarize the results of Kwon and Williams (1991), Varadhan and Williams (1985) and Williams (1985) and we prove our results in the cone; Appendix A2 contains the various auxiliary functions.

2 An inhomogeneous reverse ergodic theorem

Let E be a compact metric space, and let $Q(x,dy)$ be a sub-probability transition function on E, that is, for each $x \in E$, $Q(x,dy) \in \mathcal{M}_f(E)$ with $Q(x,E) \leq 1$ and for each $C \in \mathcal{B}(E)$, $Q(x,C)$ is Borel measurable in x. We will still denote by Q the integral operator defined by Q. In the proof of their Theorem 5.4, Bass and Pardoux (1987) show that, if Q satisfies the conditions of the Krein-Rutman theorem (Theorems 6.1 and 6.3 of Kreĭn and Rutman (1950)), then, for all $f, g \in C(E)$, $g > 0$ [2], $\{\nu_k\} \subset \mathcal{P}(E)$,

$$\lim_{k\to\infty} \frac{\int Q^k f(x)\nu_k(dx)}{\int Q^k g(x)\nu_k(dx)} = C(f,g),$$

where the constant $C(f,g)$ is independent of $\{\nu_k\}$.

Our goal is to extend this result to a sequence of compact metric spaces $E_0, E_1, E_2, \ldots$ and a sequence of sub-probability transition functions $Q_1, Q_2, \ldots$, with Q_l governing transitions from E_l to E_{l-1}, and give conditions under which (1.2), i.e.

$$\lim_{k\to\infty} \frac{\int Q_k Q_{k-1}\cdots Q_1 f(x)\nu_k(dx)}{\int Q_k Q_{k-1}\cdots Q_1 g(x)\nu_k(dx)} = C(f,g)$$

[2]The conditions of the Krein-Rutman theorem actually allow for $g \geq 0$ as long as g is not identically zero, but $g > 0$ is enough for the application in Bass and Pardoux (1987) and for our purposes as well.

with $C(f,g)$ independent of $\{\nu_k\}$, holds. We may as well take $g \equiv 1$, and we will do so in the sequel.

Lemma 2.1 *Assume*

$$\inf_{x\in E_l} Q_l(x, E_{l-1}) > 0, \quad \forall l, \tag{2.1}$$

and set, for $f \in \mathcal{C}(E_0)$,

$$T_k f(x) := \frac{Q_k Q_{k-1}\cdots Q_1 f(x)}{Q_k Q_{k-1}\cdots Q_1 1(x)}. \tag{2.2}$$

If there exists a constant $C(f)$ *such that*

$$\sup_{x\in E_k} |T_k f(x) - C(f)| \to_{k\to\infty} 0,$$

then (1.2) holds for f *and* $g = 1$ *with* $C(f,1) = C(f)$.

Remark 2.2 *Note that the operator* T_k *corresponds to a probability transition function from* E_k *to* E_0 *and can be written as*

$$T_k f(x) = P_k P_{k-1}\cdots P_1 f(x)$$

where the P_l *are the operators corresponding to the probability transition functions from* E_l *into* E_{l-1} *given by*

$$P_l(x,dy) := \frac{Q_l(x,dy)[Q_{l-1}\cdots Q_1 1(y)]}{Q_l\cdots Q_1 1(x)} = \frac{Q_l(x,dy)[Q_{l-1}\cdots Q_1 1(y)]}{\int_{E_{l-1}} Q_{l-1}\cdots Q_1 1(z) Q_l(x,dz)}. \tag{2.3}$$

Proof. Divide and multiply by $Q_k Q_{k-1}\cdots Q_1 1(x)$ inside the integral in the numerator of (1.2) (with $g = 1$). □

Lemma 2.3 *Assume* (2.1). *Define*

$$f_{l,\widetilde{x}}(x,y) := \frac{dQ_l(x,\cdot)}{d\big(Q_l(x,\cdot)+Q_l(\widetilde{x},\cdot)\big)}(y) \tag{2.4}$$

and

$$\epsilon_l(x,\widetilde{x}) := \int \left(f_{l,\widetilde{x}}(x,y) \wedge f_{l,x}(\widetilde{x},y)\right)\left(Q_l(x,dy)+Q_l(\widetilde{x},dy)\right). \tag{2.5}$$

Then

$$\begin{aligned} \|P_l(x,\cdot)-P_l(\widetilde{x},\cdot)\|_{TV} &\le 1-\epsilon_l(x,\widetilde{x})\inf_{z,y}\left(\frac{Q_{l-1}\cdots Q_1 1(y)}{Q_l\ldots Q_1 1(z)}\right) \\ &\le 1-\epsilon_l(x,\widetilde{x})\inf_{z,y}\left(\frac{Q_{l-1}\cdots Q_1 1(y)}{Q_{l-1}\ldots Q_1 1(z)}\right) \end{aligned} \tag{2.6}$$

Proof. Observe that $P_l(x, dy) << Q_l(x, dy)$ with density given by (2.3). Then

$$
\begin{aligned}
&\|P_l(x,\cdot) - P_l(\widetilde{x},\cdot)\|_{TV} \\
&\quad = \frac{1}{2}\int \left| f_{l,\widetilde{x}}(x,y)\frac{Q_{l-1}\cdots Q_1 1(y)}{Q_l\ldots Q_1 1(x)} - f_{l,x}(\widetilde{x},y)\frac{Q_{l-1}\cdots Q_1 1(y)}{Q_l\ldots Q_1 1(\widetilde{x})}\right| (Q_l(x,dy)+Q_l(\widetilde{x},dy)) \\
&\quad = 1 - \int \left(f_{l,\widetilde{x}}(x,y)\frac{Q_{l-1}\cdots Q_1 1(y)}{Q_l\ldots Q_1 1(x)}\right) \wedge \left(f_{l,x}(\widetilde{x},y)\frac{Q_{l-1}\cdots Q_1 1(y)}{Q_l\ldots Q_1 1(\widetilde{x})}\right) (Q_l(x,dy)+Q_l(\widetilde{x},dy)) \\
&\quad \le 1 - \int (f_{l,\widetilde{x}}(x,y)\wedge f_{l,x}(\widetilde{x},y)) \left(\frac{Q_{l-1}\cdots Q_1 1(y)}{Q_l\ldots Q_1 1(x)\vee Q_l\ldots Q_1 1(\widetilde{x})}\right)(Q_l(x,dy)+Q_l(\widetilde{x},dy)) \\
&\quad \le 1 - \epsilon_l(x,\widetilde{x}) \inf_{z,y}\left(\frac{Q_{l-1}\cdots Q_1 1(y)}{Q_l\ldots Q_1 1(z)}\right)
\end{aligned}
$$

The second inequality in (2.6) follows from the fact that

$$
Q_l \ldots Q_1 1(x) \le Q_l(x, E_{l-1}) \sup_z Q_{l-1}\cdots Q_1 1(z) \le \sup_z Q_{l-1}\cdots Q_1 1(z).
$$

$\square$

Theorem 2.4 *For $x, \widetilde{x} \in E_l$, let $\epsilon_l(x,\widetilde{x})$ be defined as in Lemma 2.3. Assume Q_l is not identically zero, i.e. $\sup_x Q_l(x, E_{l-1}) > 0$, for all l, and there exist $c_0 > 0$ and $\epsilon_0 > 0$ such that*

(i)

$$
\inf_{x\in E_k} Q_k\cdots Q_1 1(x) \ge c_0 \sup_{x\in E_k} Q_k\cdots Q_1 1(x), \quad \forall k,
$$

(ii)

$$
\inf_k \inf_{x,\widetilde{x}\in E_k} \epsilon_k(x,\widetilde{x}) \ge \epsilon_0.
$$

Then

$$
\sup_{x\in E_k} Q_k\cdots Q_1 1(x) > 0, \quad \forall k, \tag{2.7}
$$

and (1.2) holds for all $f \in \mathcal{C}(E_0)$ and $g = 1$, with $C(f,1) = C(f)$ given by Lemma 2.1.

Proof. First of all note that (i) above and the assumption that, for every l, Q_l is never identically zero imply, by induction, (2.7) and thus, by (i), that $\inf_{x\in E_k} Q_k\cdots Q_1 1(x) > 0$ for every k, which, in turn, implies (2.1).

Next, for $\nu \in \mathcal{P}(E_l)$, denote

$$
\nu P_l(dy) := \int_{E_l} P_l(x,dy)\nu(dx).
$$

Of course we can suppose $\epsilon_0 < 1$, $c_0 < 1$. For $\nu, \widetilde{\nu} \in \mathcal{P}(E_l)$, by Lemma 2.3, for all l,

$$
\begin{aligned}
&\|\nu P_l - \widetilde{\nu} P_l\|_{TV} = \sup_{C\in\mathcal{B}(E_{l-1})} \left| \int_{E_l}\int_{E_l} \big(P_l(x,C) - P_l(\widetilde{x},C)\big)\nu(dx)\widetilde{\nu}(d\widetilde{x})\right| \\
&\le \sup_{x,\widetilde{x}\in E_l} \|P_l(x,\cdot) - P_l(\widetilde{x},\cdot)\|_{TV} \le 1 - \epsilon_0 c_0.
\end{aligned}
$$

Then, by Lemma 5.4 of Costantini and Kurtz (2018), for $\nu, \widetilde{\nu} \in \mathcal{P}(E_k)$,

$$\begin{aligned}
&\|\nu P_k P_{k-1} \cdots P_1 - \widetilde{\nu} P_k P_{k-1} \cdots P_1\|_{TV} = \|(\nu P_k)(P_{k-1} \cdots P_1) - (\widetilde{\nu} P_k)(P_{k-1} \cdots P_1)\|_{TV} \\
&\leq \|\nu P_k - \widetilde{\nu} P_k\|_{TV} \|\nu P_{k-1} \cdots P_1 - \widetilde{\nu} P_{k-1} \cdots P_1\|_{TV} \\
&\leq (1 - \epsilon_0 c_0) \|\nu P_{k-1} \cdots P_1 - \widetilde{\nu} P_{k-1} \cdots P_1\|_{TV},
\end{aligned}$$

and, by iterating,

$$\|\nu P_k P_{k-1} \cdots P_1 - \widetilde{\nu} P_k P_{k-1} \cdots P_1\|_{TV} \leq (1 - \epsilon_0 c_0)^k.$$

In particular, for each $f \in \mathcal{C}(E_0)$, for an arbitrary $\{x_k\}$, $x_k \in E_k$ for each k,

$$|T_{k+l} f(x_{k+l}) - T_k f(x_k)| \leq \|(\delta_{x_{k+l}} P_{k+l} \cdots P_{k+1}) P_k \cdots P_1 - \delta_{x_k} P_k \cdots P_1\|_{TV} \|f\| \leq (1 - \epsilon_0 c_0)^k \|f\|,$$

so that $\{T_k f(x_k)\}$ is a Cauchy sequence. If $C(f)$ is its limit, we get, in an analogous manner,

$$\sup_{x \in E_k} |T_k f(x) - C(f)| \leq (1 - \epsilon_0 c_0)^k \|f\| + |T_k f(x_k) - C(f)|,$$

which yields the assertion by Lemma 2.1. □

3 Existence and uniqueness for reflecting diffusions in a domain with one singular point

3.1 Formulation of the problem and preliminaries

We consider a simply connected domain $D \subseteq \mathbb{R}^d$ that has a smooth boundary except at a single point, which we will take to be the origin, and that in a neighborhood of the singular point looks like a cone. More precisely we assume the following condition (d_H denotes the Hausdorff distance).

Condition 3.1

(i) $\partial D - \{0\}$ *is of class* $\mathcal{C}^1$. *There exists a nonempty domain,* $\mathcal{S}$, *in the unit sphere,* S^{d-1}, *such that, setting*

$$\mathcal{K} := \{x : x = rz, \, z \in \mathcal{S}, \, r > 0\},$$

it holds, for r *less or equal than some positive constant* r_D,

$$d_H(\overline{D} \cap \partial B_r(0), \overline{\mathcal{K}} \cap \partial B_r(0)) \leq c_D r^2,$$

$$d_H(\partial D \cap \partial B_r(0), \partial \mathcal{K} \cap \partial B_r(0)) \leq c_D r^2,$$

$$d_H((\overline{D} - \overline{\mathcal{K}}) \cap \partial B_r(0), \partial \mathcal{K} \cap \partial B_r(0)) \leq c_D r^2,$$

$$d_H((\overline{\mathcal{K}} - \overline{D}) \cap \partial B_r(0), \partial D \cap \partial B_r(0)) \leq c_D r^2,$$

and, denoting by $\bar{z}$ *the closest point to* $\frac{x}{|x|}$ *on* $\partial \mathcal{S}$ *and by* $n(x)$, $n^{\mathcal{K}}(x)$ *the unit inward normal to* $\overline{D}$ *and to* $\overline{\mathcal{K}}$ *at* $x \neq 0$, *respectively,*

$$|n(x) - n^{\mathcal{K}}(|x|\bar{z})| \leq c_D |x|, \quad \forall x \in \partial D - \{0\}.$$

D *is a bounded domain.*

(ii) For $d \geq 3$, the boundary $\partial\mathcal{S}$ of $\mathcal{S}$ in S^{d-1} is of class $\mathcal{C}^3$.

Remark 3.2 *Clearly Condition 3.1(i) implies that, for every $x \in \partial D$, $|x| \leq r_D$,*

$$d(\frac{x}{|x|}, \partial\mathcal{S}) \leq c_D|x|,$$

and hence, by the smoothness of $\partial\mathcal{S}$ (Condition 3.1(ii)), for r_D sufficiently small, there is a unique $\bar{z} \in \partial\mathcal{S}$ such that

$$|\frac{x}{|x|} - \bar{z}| = d(\frac{x}{|x|}, \partial\mathcal{S}).$$

We assume the following on the directions of reflection.

Condition 3.3

(i) $g : \mathbb{R}^d - \{0\} \to \mathbb{R}^d$ is a locally Lipschitz vector field, of unit length on ∂D, such that

$$\inf_{x \in \partial D - \{0\}} g(x) \cdot n(x) > 0.$$

There exists a unit vector field $\bar{g} : \partial\mathcal{S} \to \mathbb{R}^d$ such that, for $x \in \partial D$ and $\bar{z}$ the closest point to $\frac{x}{|x|}$ on $\partial\mathcal{S}$, it holds, for $|x| \leq r_D$,

$$|g(x) - \bar{g}(\bar{z})| \leq c_g|x|.$$

(ii) Define $\bar{g}$ on $\partial\mathcal{K} - \{0\}$ by

$$\bar{g}(x) := \bar{g}(\frac{x}{|x|}).$$

$\bar{g}$ satisfies

$$\inf_{x \in \partial\mathcal{K} - \{0\}} \bar{g}(x) \cdot n^{\mathcal{K}}(x) > 0.$$

For $d \geq 3$, $\bar{g}$ is of class $\mathcal{C}^2$. For $x \in \partial\mathcal{K} - \{0\}$, denoting by $n^r(x) := \frac{x}{|x|}$, the radial unit vector, $\frac{\bar{g} \cdot n^r}{\bar{g} \cdot n^{\mathcal{K}}}$ is of class $\mathcal{C}^3$ and $\frac{\bar{g} - \bar{g} \cdot n^r n^r - \bar{g} \cdot n^{\mathcal{K}} n^{\mathcal{K}}}{\bar{g} \cdot n^{\mathcal{K}}}$ is of class $\mathcal{C}^4$.

(iii) For $x \in \partial D - \{0\}$, let $G(x) := \{\eta g(x),\ \eta \geq 0\}$, and let $G(0)$ be the closed, convex cone generated by $\{\bar{g}(z),\ z \in \partial\mathcal{S}\}$. Assume

$$G(0) \cap \mathcal{K} \neq \emptyset.$$

(iv) Let $N(0)$ denote the normal cone at the origin for $\overline{D}$, that is,

$$N(0) := \{n \in \mathbb{R}^d : \liminf_{x \in \overline{D} - \{0\},\, |x| \to 0} n \cdot \frac{x}{|x|} \geq 0\}.$$

There exists a unit vector $e \in N(0)$ such that

$$\inf_{g \in G(0),\, |g| = 1} e \cdot g = c_e > 0.$$

Remark 3.4 *By Condition 3.1(i), $N(0)$ is also the normal cone at the origin for $\overline{\mathcal{K}}$, i.e.*

$$N(0) = \{n \in \mathbb{R}^d : n \cdot x \geq 0, \forall x \in \overline{\mathcal{K}}\}.$$

$N(0)$ is a closed convex cone.

Condition 3.3(iv) implies that $N(0) \neq \emptyset$, hence $\overline{\mathcal{K}}$ is contained in a closed halfspace.

Note that we are not assuming that the interior of $N(0)$ is nonempty, therefore $\overline{\mathcal{K}}$ is allowed to contain full straight lines. In particular we are allowing ∂D to be $\mathcal{C}^2$, but g to be discontinuous at the origin.

If the interior of $N(0)$ is nonempty, we can assume, w.l.o.g., that $e \in \overset{\circ}{N}(0)$, and hence that there exists $c'_e > 0$ such that, for $x \in \overline{D} - \{0\}$, $|x| \leq r_D$,

$$x \cdot e \geq c'_e |x|.$$

Remark 3.5 *Conditions 3.1(ii) and 3.3(ii) are the assumptions of Kwon and Williams (1991), which we need because we will exploit some of their results.*

Reflecting diffusions are often characterized as solutions of stochastic differential equations. Assume the following.

Condition 3.6

(i) $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are Lipschitz continuous.

(ii) $\sigma(0)$ is non singular.

Definition 3.7 *A stochastic process X is a solution of the* stochastic differential equation with reflection *in $\overline{D}$ with coefficients b and σ and direction of reflection g, if there exist a standard Brownian motion W, an a.s. continuous, non decreasing process λ and an a.s. measurable process γ, all defined on the same probability space as X, such that $W(t+\cdot)-W(t)$ is independent of $\mathcal{F}_t^{X,W,\lambda,\gamma}$ for all $t \geq 0$ and the equation*

$$\begin{gathered} X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(s)\, d\lambda(s), \quad t \geq 0, \\ \gamma(t) \in G(X(t)), \quad |\gamma(t)| = 1, \quad d\lambda - a.e., \quad t \geq 0, \\ X(t) \in \overline{D}, \quad \lambda(t) = \int_0^t \mathbf{1}_{\partial D}(X(s))d\lambda(s), \quad t \geq 0, \end{gathered} \tag{3.1}$$

is satisfied a.s..

Given an initial distribution $\mu \in \mathcal{P}(\overline{D})$, weak uniqueness *or* uniqueness in distribution *holds if all solutions of (3.1) with $P\{X(0) \in \cdot\} = \mu$ have the same distribution on $\mathcal{C}_{\overline{D}}[0,\infty)$.*

A stochastic process $\widetilde{X}$ (for example a solution of an appropriate martingale problem or submartingale problem) is a weak solution of (3.1) *if there is a solution X of (3.1) such that $\widetilde{X}$ and X have the same distribution.*

We denote by A the operator

$$\mathcal{D}(A) := \mathcal{C}^2(\overline{D}), \quad Af(x) := b(x) \cdot \nabla f(x) + \frac{1}{2}\mathrm{tr}((\sigma\sigma^T)(x)D^2 f(x)). \tag{3.2}$$

Remark 3.8 *Since $\sigma(0)$ is non singular, it is easy to check that X is a solution to (3.1) if and only if $\sigma^{-1}(0)X$ is a solution of (3.1), with the appropriate coefficients, in the corresponding domain $\sigma^{-1}(0)D$, with vector field of the directions of reflection $\frac{\sigma^{-1}(0)g\circ\sigma(0)}{|\sigma^{-1}(0)g\circ\sigma(0)|}$, and the new domain and vector field of directions of reflection satisfy Condition 3.1 and Condition 3.3. Therefore, without loss of generality, we will take, from now on,*

$$\sigma(0) = I.$$

Conditions 3.1, 3.3 and 3.6 will be our standing assumptions.

As we will see in the next subsections, it turns out to be more convenient to characterize a reflecting diffusion process as the *natural solution* of the *constrained martingale problem* (see Kurtz (1990) and Kurtz (1991)) defined below; however, Theorem 3.18 below shows that the two characterizations are equivalent.

Let

$$\begin{aligned} U \quad &:= \quad S^{d-1}, \\ \Xi := \{(x,u) \in \partial D \times U \quad &: \quad u \in G(x)\}, \\ B : \mathcal{D}(B) := \mathcal{C}^2(\overline{D}) \;\to\; \mathcal{C}(\Xi), \qquad & Bf(x,u) := \nabla f(x) \cdot u \end{aligned} \tag{3.3}$$

Note that Ξ is closed by Condition 3.3(i).

Define $\mathcal{L}_U$ to be the space of measures μ on $[0,\infty) \times U$ such that $\mu([0,t] \times U) < \infty$ for all $t > 0$. $\mathcal{L}_U$ is topologized so that $\mu_n \in \mathcal{L}_U \to \mu \in \mathcal{L}_U$ if and only if

$$\int_{[0,\infty)\times U} f(s,u)\mu_n(ds \times du) \to \int_{[0,\infty)\times U} f(s,u)\mu(ds \times du)$$

for all continuous f with compact support in $[0,\infty) \times U$. It is possible to define a metric on $\mathcal{L}_U$ that induces the above topology and makes $\mathcal{L}_U$ into a complete, separable metric space. Also define $\mathcal{L}_\Xi$ in the same way.

We will say that an $\mathcal{L}_U$-valued random variable Λ_1 is adapted to a filtration $\{\mathcal{G}_t\}$ if

$$\Lambda_1([0,\cdot] \times C) \text{ is } \{\mathcal{G}_t\} - \text{adapted}, \quad \forall C \in \mathcal{B}(U).$$

We define an adapted $\mathcal{L}_\Xi$-valued random variable analogously.

Definition 3.9 *(Kurtz (1991)) Let A, U, Ξ and B be as in (3.2) and (3.3). A process X in $\mathcal{D}_{\overline{D}}[0,\infty)$ is a solution of the* constrained (local) martingale problem *for (A, D, B, Ξ) if there exists a random measure Λ with values in $\mathcal{L}_\Xi$ and a filtration $\{\mathcal{F}_t\}$ such that X and Λ are $\{\mathcal{F}_t\}$-adapted and for each $f \in \mathcal{C}^2(\overline{D})$,*

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_{[0,t]\times\Xi} Bf(x,u)\Lambda(ds \times dx \times du) \tag{3.4}$$

is a $\{\mathcal{F}_t\}$-local martingale. By the continuity of f, we may assume, without loss of generality, that $\{\mathcal{F}_t\}$ is right continuous.

Remark 3.10 *Since $f(x) := x_i$ $i = 1, ..., d$, belongs to $\mathcal{D}(A) = \mathcal{D}(B)$, every solution of the constrained martingale problem for (A, D, B, Ξ) is a semimartingale.*

An effective way of constructing solutions of a constrained martingale problem is by time-changing solutions of the corresponding controlled martingale problem which is a "slowed down" version of the constrained martingale problem.

Definition 3.11 *(Kurtz (1991)) Let A, U, Ξ and B be as in* (3.2) *and* (3.3). *$(Y, \lambda_0, \Lambda_1)$ is a solution of the* controlled martingale problem *for (A, D, B, Ξ), if Y is a process in $\mathcal{D}_{\overline{D}}[0, \infty)$, λ_0 is nonnegative and nondecreasing, Λ_1 is a random measure with values in $\mathcal{L}_U$ such that*

$$\lambda_1(t) := \Lambda_1([0,t] \times U) = \int_{[0,t]\times U} \mathbf{1}_{\Xi}(Y(s), u)\Lambda_1(ds \times du), \tag{3.5}$$

$$\lambda_0(t) + \lambda_1(t) = t,$$

and there exists a filtration $\{\mathcal{G}_t\}$ such that Y, λ_0, and Λ_1 are $\{\mathcal{G}_t\}$-adapted and

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_{[0,t]\times U} Bf(Y(s), u)\Lambda_1(ds \times du) \tag{3.6}$$

is a $\{\mathcal{G}_t\}$-martingale for all $f \in \mathcal{C}^2(\overline{D})$. We can assume, without loss of generality, that $\{\mathcal{G}_t\}$ is right continuous.

Remark 3.12 *It can be easily verified (e.g. by Proposition 3.10.3 of Ethier and Kurtz (1986)) that, for every solution of the controlled martingale problem for (A, D, B, Ξ), Y is continuous.*

Definition 3.13 *Let A, U, Ξ and B be as in* (3.2) *and* (3.3). *A solution of the constrained martingale problem for (A, D, B, Ξ) is called* natural *if, for some solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, D, B, Ξ) with (right continuous) filtration $\{\mathcal{G}_t\}$,*

$$X(t) = Y(\lambda_0^{-1}(t)), \quad \mathcal{F}_t = \mathcal{G}_{\lambda_0^{-1}(t)}, \quad \lambda_0^{-1}(t) = \inf\{s : \lambda_0(s) > t\}, \qquad t \geq 0$$
$$\Lambda([0,t] \times C) = \int_{[0,\lambda_0^{-1}(t)]\times U} \mathbf{1}_C(Y(s), u)\Lambda_1(ds \times du), \quad C \in \mathcal{B}(\Xi). \tag{3.7}$$

Given a a solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, D, B, Ξ), the time changed process X defined by 3.7 will not always be a solution of the corresponding constrained martingale problem. In fact it may be impossible to stop (3.6), after the time change by λ_0^{-1}, in such a way that the stopped process is a local martingale. Conditions under which it is possible are given in Costantini and Kurtz (2019) (Corollary 3.9) and the following lemma guarantees that they are satisfied under our standing assumptions.

Lemma 3.14 *There exists a function $F \in \mathcal{C}^2(\overline{D})$ such that*

$$\inf_{x\in\partial D - \{0\}} \nabla F(x) \cdot g(x) := c_F > 0.$$

Proof. See Appendix A2. $\square$

Proposition 3.15 *Let A, U, Ξ and B be as in* (3.2) *and* (3.3) *and assume Conditions 3.1, 3.3 and 3.6.*

For every solution of the controlled martingale problem for (A, D, B, Ξ)*, the time changed process X defined by 3.7 is a natural solution of the corresponding constrained martingale problem and* (3.4) *is a martingale.*

Proof. By Lemma 3.14 and Lemma 3.1 of Costantini and Kurtz (2019), $\lambda_0^{-1}(t)$ is a.s. finite for every $t \geq 0$ and, after the time change by λ_0^{-1}, (3.6) is a martingale. □

The following lemma yields the equivalence between the definitions of a reflecting diffusion as solution of an SDER and as solution of a constrained martingale problem. It will also be used in Section 3.2.

Lemma 3.16 *For every solution* $(Y, \lambda_0, \Lambda_1)$ *of the controlled martingale problem for* (A, D, B, Ξ)*, $\lambda_0(t) > 0$ for all $t > 0$, a.s..*

Moreover, λ_0 is strictly increasing, a.s..

Proof. The first assertion follows essentially from Condition 3.3(iv) and Remark 3.12. The proof is analogous to that of Lemma 6.8 of Costantini and Kurtz (2019) and Lemma 3.1 of Dai and Williams (1996). The second assertion follows from Lemma 3.4 of Costantini and Kurtz (2019). □

Remark 3.17 *It follows from Remark 3.12 and Lemma 3.16 that every natural solution of the constrained martingale problem for* (A, D, B, Ξ) *is a.s. continuous and so is the corresponding process* $\Lambda([0, \cdot] \times \Xi)$.

Theorem 3.18 *Let A, U, Ξ and B be as in* (3.2) *and* (3.3) *and assume Conditions 3.1, 3.3 and 3.6.*

Every solution of (3.1) *is a natural solution of the constrained martingale problem for* (A, D, B, Ξ).

Every natural solution of the constrained martingale problem for (A, D, B, Ξ) *is a weak solution of* (3.1).

Proof. The proof is the same as for Theorem 6.12 of Costantini and Kurtz (2019). It relies essentially on Lemma 3.16 and Remark 3.17. □

Remark 3.19 *Proposition 3.15 could have been proved, alternatively, by Lemma 3.16 and Lemmas 3.3 and 3.4 of Costantini and Kurtz (2019).*

We conclude this section with two important properties of a natural solution X of the constrained martingale problem for (A, D, B, Ξ). For $\delta > 0$, define

$$\tau^\delta := \inf\{\{t \geq 0 : |X(t)| = \delta.\} \tag{3.8}$$

Lemma 3.20 *There exists $\bar{\delta} > 0$, $\bar{c} > 0$, depending only on the data of the problem, such that, for $\delta \leq \bar{\delta} \wedge r_D$, for every natural solution X of the constrained martingale problem for* (A, D, B, Ξ) *starting at* 0,

$$\mathbb{E}^0[\tau^\delta] \leq \bar{c}\delta^2.$$

Proof. The assertion follows essentially from Condition 3.3(iv). The proof is analogous to that of Lemma 4.2 of Costantini and Kurtz (2018) and Lemma 6.4 of Taylor and Williams (1993). $\square$

Lemma 3.21 *For every natural solution X of the constrained martingale problem for $(A, \overline{D}, B, \Xi)$,*

$$\int_0^\infty \mathbf{1}_{\{0\}}(X(t))\, dt = 0, \quad a.s..$$

Proof. The proof uses the same argument as Lemma 2.1 of Taylor and Williams (1993). Fix an arbitrary unit vector v. Then, by Remark 3.17,

$$m(t) := v \cdot X(t) - v \cdot X(0) - \int_0^t v \cdot b(X(s))\, ds - \int_{[0,t]\times\Xi} v \cdot u\, \Lambda(ds \times du)$$

is a continuous semimartingale with

$$[m, m](t) = \int_0^t |\sigma(X(s))^T v|^2 ds.$$

Therefore, by Tanaka's formula (see, e.g., Protter (2004), Corollary 1, Chapter IV, Section 7), for each $t > 0$,

$$\int_0^t \mathbf{1}_{\{0\}}(X(s) \cdot v)\, |\sigma(X(s))^T v|^2 ds = \int_{\mathbb{R}} \mathbf{1}_{\{0\}}(a)\, L_t(a) da = 0, \quad a.s.,$$

$L_t(a)$ being the local time of m at a. Hence the set of times $\{s \leq t : X(s) \cdot v = 0 \text{ and } |\sigma(X(s))^T v| \neq 0\}$ has zero Lebesgue measure, a.s., which yields the assertion by By Condition 3.6(ii). $\square$

3.2 Existence

In this subsection we show that there exists a strong Markov, natural solution of the constrained martingale problem for (A, D, B, Ξ), and hence, by Theorem 3.18, of the stochastic differential equation with reflection 3.1. The strong Markov property will be crucial in our argument to prove uniqueness of the solution.

Theorem 3.22 *Under Conditions 3.1, 3.3 and 3.6, for each $\nu \in \mathcal{P}(\overline{D})$, there exists a strong Markov, natural solution of the constrained martingale problem for (A, D, B, Ξ), with initial distribution ν.*

Proof. We will first construct a solution of the controlled martingale problem. Let $\{\delta_k\}$ be a strictly decreasing sequence of positive numbers converging to zero, and let $\{D^k\}$ be a sequence of bounded domains with $\mathcal{C}^1$ boundary such that $D^k \subset D^{k+1} \subset D$, $\overline{D^k} \cap B_{\delta_k}(0)^c = \overline{D} \cap B_{\delta_k}(0)^c$ and $\overline{D^k} \cap B_{\delta_k}(0) \subset D^{k+1}$. Also let $g^k : \mathbb{R}^d - \{0\} \to \mathbb{R}^d$, $k \in \mathbb{N}$, be a locally Lipschitz vector field, of unit length on ∂D^k, such that $g^k(x) = g(x)$ for $x \in \partial D \cap \Big(B_{\delta_k}(0)\Big)^c$ and that, denoting by $n^k(x)$ the unit, inward normal at $x \in \partial D^k$, it holds $\inf_{x\in\partial D^k} g^k(x) \cdot n^k(x) > 0$. Then we know, by the results of Dupuis and Ishii (1993), that, given a standard Brownian motion, W, for each $\overline{D^k}$-valued random variable ξ_0^k, independent of W, there exists a unique stochastic process ξ^k for which there is a nondecreasing process l^k such that (ξ^k, l^k) satisfies (3.1) in $\overline{D^k}$ with direction of reflection g^k. The sequence of stopping times $\{\theta^k\}$,

$$\theta^k := \inf\{t \geq 0 : \, \xi^k(t) \in \partial D^k \cap B_{\delta_k}(0)\},$$

is strictly increasing and, setting

$$\theta := \lim_{k\to\infty} \theta^k,$$

for each random variable ξ_0 with compact support in $\overline{D} - \{0\}$, independent of W, we can define a pair of stochastic processes ξ and l that satisfies

$$\begin{aligned} \xi(t) &= \xi_0 + \int_0^t b(\xi(s))ds + \int_0^t \sigma(\xi(s))dW(s) + \int_0^t g(\xi(s))dl(s), \\ & \qquad \xi(t) \in \overline{D} - \{0\}, \quad \int_0^t \mathbf{1}_{\partial D - \{0\}}(\xi(s))dl(s) = l(t), \quad a.s., \end{aligned} \tag{3.9}$$

for $0 \leq t < \theta$. We will now show that, on the set $\{\theta < \infty\}$ it holds, a.s.,

$$\sup_{0\leq t<\theta} l(t) < \infty, \qquad \lim_{t\to\theta^-} \xi(t) = 0. \tag{3.10}$$

In fact, let F be the function of Lemma 3.14. Then, by Ito's formula, on the set $\{\theta < \infty\}$, we have

$$\begin{aligned} l(\theta^k) &\leq \frac{1}{c_F}\left\{2\|F\| + \|AF\|\,\theta_k + \left| \int_0^{\theta^k} \nabla F(\xi(s)^T \sigma(\xi(s))dW(s) \right| \right\} \\ &= \frac{1}{c_F}\left\{2\|F\| + \|AF\|\theta_k + \left| \int_0^{\theta^k} \mathbf{1}_{\{s<\theta\}} \nabla F(\xi(s)^T \sigma(\xi(s))dW(s) \right| \right\}. \end{aligned}$$

Since the process $\mathbf{1}_{\{s<\theta\}}\nabla F(\xi(s)^T\sigma(\xi(s))$ is predictable and bounded, the process $\int_0^t \mathbf{1}_{\{s<\theta\}}\nabla F(\xi(s)^T\sigma(\xi(s))dW(s)$ is a.s. continuous. Therefore, on the set $\{\theta < \infty\}$, the limit $\lim_{k\to\infty} \int_0^{\theta^k} \mathbf{1}_{\{s<\vartheta\}}\nabla F(\xi(s)^T\sigma(\xi(s))dW(s)$ exists and is finite a.s.. This yields the first assertion in (3.10) and allows to define, on the set $\{\theta < \infty\}$, $l(\theta) := \sup_{0\leq t<\theta} l(t)$. The second assertion follows by observing that, on the set $\{\theta < \infty\}$, both $l(t)$ and $\int_0^t \mathbf{1}_{\{s<\theta\}}\nabla F(\xi(s)^T\sigma(\xi(s))dW(s)$ are continuous on $[0, \theta]$ a.s..

By (3.10), on the set $\{\theta < \infty\}$, the solution of (3.9) is uniquely defined up to θ included.

Now let g^0 be a unit vector in $G(0) \cap \mathcal{K}$ (Condition 3.3 (iii)). Then, by Condition 3.1(i), for ρ small enough $\rho g^0 \in G(0) \cap D$. Let $\{\rho_n\}$ be a decreasing sequence of positive numbers such that $\rho_n \to 0$ and $\rho_n g^0 \in G(0) \cap D$ for all n. Consider the stochastic differential equation with reflection

$$\begin{aligned} X^n(t) &= X_0^n + \int_0^t b(X^n(s))ds + \int_0^t \sigma(X^n(s))dW(s) + \int_0^t g(X^n(s))dl^n(s) + \rho_n g^0 L^n(t), \\ & \quad X^n(t) \in \overline{D} - \{0\}, \quad l^n \text{ non decreasing}, \quad \int_0^t \mathbf{1}_{\partial D - \{0\}}(X^n(s))dl^n(s) = l^n(t), \\ & \quad L^n(t) = \#\{s \leq t : X^n(s^-) = 0\}, \end{aligned} \tag{3.11}$$

where # denotes cardinality, X_0^n is a random variable with compact support in $\overline{D}-\{0\}$ and W is a standard Brownian motion independent of X_0^n. Existence of X^n follows from the existence of the solution of (3.9) up to (and included, if finite) the first time the process hits 0.

Define

$$\lambda_0^n(t) := \inf\{s : s + l^n(s) + \rho^n L^n(s) > t\},$$

$$\Lambda_1^n([0,t]\times C) := \int_0^t \mathbf{1}_C(g(X^n(\lambda_0^n(s)))\, dl^n(\lambda_0^n(s))) + \mathbf{1}_C(g^0)\rho_n L^n(\lambda_0^n(s)),$$

$$B^n f(x,u) := u\cdot\nabla f(x)\mathbf{1}_{\partial D-\{0\}}(x) + (\rho_n)^{-1}[f(x+\rho_n u) - f(x)]\mathbf{1}_{\{0\}}(x),$$

and $Y^n(t) := X^n(\lambda_0^n(t))$. Then for each $f \in \mathcal{C}^2(\bar{D})$,

$$f(Y^n(t)) - f(Y^n(0)) - \int_0^t Af(Y^n(s))d\lambda_0^n(s) - \int_{[0,t]\times U} B^n f(Y^n(s^-),u)\Lambda_1^n(ds\times du)$$

is a martingale with respect to $\{\mathcal{F}_t^{Y^n,\lambda_0^n,\Lambda_1^n}\}$. $(Y^n, \lambda_0^n, \Lambda_1^n)$ is not a solution of the controlled martingale problem for (A, D, B^n, Ξ) in the precise sense of Costantini and Kurtz (2019) and Kurtz (1991), because $B^n f$ is not continuous on Ξ and because we can only say

$$t \le \lambda_0^n(t) + \Lambda_1^n([0,t]\times U) \le t + \rho_n.$$

However, the same relative compactness arguments apply (see, for example, Lemma 2.8 of Costantini and Kurtz (2019)) and, if the law of $Y^n(0)$ converges to ν, any limit point of $\{(Y^n, \lambda_0^n, \Lambda_1^n)\}$ will be a solution of the controlled martingale problem for (A, D, B, Ξ) with initial distribution ν.

Then, taking into account Lemma 3.14, Lemma 3.1 of Costantini and Kurtz (2019), Condition 3.5 of Costantini and Kurtz (2019) is satisfied, and the assertion follows from Lemma 3.14, Lemma 3.16 and Corollary 4.12 a) of Costantini and Kurtz (2019).

□

3.3 Outline of the proof of uniqueness

Our approach follows the general outline of Kwon and Williams (1991), but in order to deal with curved boundaries, general diffusions, and varying directions of reflection, we replace some analytical building blocks of Kwon and Williams (1991) by corresponding probabilistic results. In particular, we replace the application of the Krein-Rutman theorem in Kwon and Williams (1991) by the inhomogeneous ergodic theorem of Section 2. In turn, some of the estimates we need to apply our probabilistic results exploit some analytical results of Kwon and Williams (1991) and a result proved in Appendix A1, together with the coupling result of Lemma 5.3 of Costantini and Kurtz (2018). Another essential ingredient of our arguments is that, in order to prove uniqueness of the solution of the constrained martingale problem for $(A, \overline{D}, B, \Xi)$, it is enough to prove uniqueness among strong Markov, natural solutions (Costantini and Kurtz (2019)).

Recall that we are assuming Conditions 3.1, 3.3 and 3.6 throughout this section.

Fix $0 < \delta^* \leq \bar{\delta}$, where $\bar{\delta}$ is as in Lemma 3.20, and let

$$D_n := \overline{D} \cap B_{\delta^* 2^{-2n}}, \quad E_n := \{x \in \overline{D} : |x| = \delta^* 2^{-2n}\}, \quad n \geq 1. \tag{3.12}$$

Let X be a solution of the constrained martingale problem for (A, D, B, Ξ). Define recursively

$$\vartheta = \vartheta_0^n := \inf\{t \geq 0 : X(t) = 0\}, \qquad \tau^n := \inf\{t \geq 0 : X(t) \in E_n\}, \qquad n \geq 0,$$

$$\tau_l^n := \inf\{t > \vartheta_{l-1}^n : X(t) \in E_n\}, \quad \vartheta_l^n := \inf\{t > \tau_l^n : X(t) = 0\}, \qquad l \geq 1,\, n \geq 0, \tag{3.13}$$

In order to simplify the notation, we set

$$\tau^0 = \tau, \quad \tau_l^0 = \tau_l, \quad \vartheta_l^0 = \vartheta_l. \tag{3.14}$$

Lemma 3.23 *Suppose that the hitting distributions $\{\mu_n\}$ defined by*

$$\mu_n(C) := \mathbb{P}\{X(\tau^n) \in C\}, \quad C \in \mathcal{B}(E_n),$$

are the same for all strong Markov, natural solutions of the constrained martingale problem for (A, D, B, Ξ) starting at 0.

Then, for each $\nu \in \mathcal{P}(\overline{D})$, there is a unique natural solution of the constrained martingale problem for (A, D, B, Ξ) with initial distribution ν.

Proof. Lemma 3.16 allows to apply Corollary 4.13 of Costantini and Kurtz (2019). Therefore it is enough to prove uniqueness among strong Markov, natural solutions of the constrained martingale problem for (A, D, B, Ξ). Let X be such a solution with initial distribution ν.

Set $\tau_0^n := 0$. For $\eta > 0$ and $f \in \mathcal{C}(\overline{D})$ that vanishes in a neighborhood of the origin, define, for each $n \geq 0$,

$$R_\eta^n f := \mathbb{E}\Big[\int_0^\vartheta e^{-\eta t} f(X(t))dt\Big] + \mathbb{E}\Big[\sum_{l=1}^\infty \mathbb{E}\Big[\prod_{m=0}^{l-1} e^{-\eta(\vartheta_m^n - \tau_m^n)} \int_{\tau_l^n}^{\vartheta_l^n} e^{-\eta(t-\tau_l^n)} f(X(t))dt\Big].$$

The hypothesis ensures that the distribution of $X(\tau_l^n)$ is μ^n for all l. Then, by the strong Markov property, the factors in the second expectation are independent, with distributions determined by the initial distribution ν and by the unique distribution of $X^n(\cdot \wedge \vartheta^n)$, where X^n is a solution of the constrained martingale problem with initial distribution μ^n and ϑ^n is the first time X^n hits zero. Consequently, each term on the right side is uniquely determined. The independence implies also that the series is convergent for any f.

Let $\mathcal{T}^n := [0, \vartheta] \cup \cup_{l=1}^\infty (\tau_l^n, \vartheta_l^n]$. Then $R_\eta^n f$ can be written as

$$R_\eta^n f = \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\mathcal{T}^n}(t) e^{-\eta \int_0^t \mathbf{1}_{\mathcal{T}^n}(s)ds} f(X(t))dt\right] = \mathbb{E}\left[\int_0^\infty e^{-\eta \int_0^t \mathbf{1}_{\mathcal{T}^n}(s)ds} f(X(t))dt\right],$$

where the latter equality holds for n large enough, depending only on f.

Now we have

$$\int_0^\infty \mathbf{1}_{\mathcal{T}^n}(s)ds = \vartheta + \sum_{l=1}^\infty (\vartheta_l^n - \tau_l^n) = \infty, \quad a.s.,$$

because the random variables in the sum in the right side are positive i.i.d. random variables. Hence, for each $t_0 \geq 0$, for n large enough, (depending only on f),

$$\begin{aligned} \int_{t_0}^{\infty} e^{-\eta \int_0^t \mathbf{1}_{\mathcal{T}^n}(s)ds} |f(X(t))|dt &= \int_{t_0}^{\infty} \mathbf{1}_{\mathcal{T}^n}(t) e^{-\eta \int_0^t \mathbf{1}_{\mathcal{T}^n}(s)ds} |f(X(t))|dt \qquad (3.15) \\ &\leq \|f\| \frac{1}{\eta} e^{-\eta \int_0^{t_0} \mathbf{1}_{\mathcal{T}^n}(s)ds}, \qquad a.s.. \qquad (3.16) \end{aligned}$$

On the other hand, by Lemma 3.21, for each $t \geq 0$,

$$e^{-\eta \int_0^t \mathbf{1}_{\mathcal{T}^n}(s)ds} \to e^{-\eta t}, \qquad a.s.$$

Therefore, a.s., the sequence $\left\{ e^{-\eta \int_0^t \mathbf{1}_{\mathcal{T}^n}(s)ds} f(X(t)) \right\}$ converges to $e^{-\lambda t} f(X(t))$ and is uniformly integrable, so that

$$\int_0^{\infty} e^{-\eta \int_0^t \mathbf{1}_{\mathcal{T}^n}(s)ds} f(X(t))dt \to_{n\to\infty} \int_0^{\infty} e^{-\eta t} f(X(t))dt, \quad a.s.,$$

and, again by (3.16), with $t = t_0$,

$$R_\eta^n f = \mathbb{E}\left[\int_0^{\infty} e^{-\eta \int_0^t \mathbf{1}_{\mathcal{T}^n}(s)ds} f(X(t))dt \right] \to_{n\to\infty} \mathbb{E}\left[\int_0^{\infty} e^{-\eta t} f(X(t))dt \right] := R_\eta f.$$

Since the class of continuous functions on $\overline{D}$ that vanish in a neighborhood of the origin is separating, it follows that the finite dimensional distributions of X are uniquely determined. □

Theorem 3.24 *Under Conditions 3.1, 3.3 and 3.6, for each $\nu \in \mathcal{P}(\overline{D})$, there is a unique natural solution to the constrained martingale problem for (A, D, B, Ξ) with initial distribution ν.*

Proof. We only need to show that the assumption of Lemma 3.23 is verified.

Let X be a strong Markov, natural solution of the constrained martingale problem for (A, D, B, Ξ) with $X(0) = 0$. Then

$$\begin{aligned} \mathbb{E}^0[f(X(\tau))] &= \mathbb{E}^0[f(X(\tau), \tau < \vartheta] + \mathbb{E}^0[f(X(\tau)), \tau > \vartheta] \\ &= \mathbb{E}^0[\mathbb{E}[f(X(\tau), \tau < \vartheta | \mathcal{F}_{\tau^1}]] + \mathbb{E}^0[\mathbb{E}[f(X(\tau)) | \mathcal{F}_\vartheta], \tau > \vartheta] \\ &= \mathbb{E}^0[\mathbb{E}^{X(\tau^1)}[f(X(\tau), \tau < \vartheta]] + \mathbb{E}^0[f(X(\tau))]\, \mathbb{E}^0[\mathbb{E}^{X(\tau^1)}\{\tau > \vartheta\}] \end{aligned}$$

where the last but one equality follows from the strong Markov property. This identity gives

$$\mathbb{E}^0[f(X(\tau))] = \frac{\mathbb{E}^0[\mathbb{E}^{X(\tau^1)}[f(X(\tau), \tau < \vartheta]]}{\mathbb{E}^0[\mathbb{E}^{X(\tau^1)}\{\tau < \vartheta\}]} \qquad (3.17)$$

Define

$$\begin{aligned} Q_1 f(x) &:= \mathbb{E}^x[f(X(\tau)), \tau < \vartheta], \quad x \in E_1, \\ Q_k f(x) &:= \mathbb{E}^x[f(X(\tau^{k-1})), \tau^{k-1} < \vartheta], \quad x \in E_k, \quad k \geq 2. \end{aligned} \qquad (3.18)$$

Note that each Q_k is uniquely determined, by the uniqueness of the distribution until the origin is hit. Iterating (3.17), we have

$$\mathbb{E}^0[f(X(\tau))] = \frac{\mathbb{E}^0[Q_k \cdots Q_1 f(X(\tau^k)]}{\mathbb{E}^0[Q_k \cdots Q_1 1(X(\tau^k)]}.$$

Then, if Theorem 2.4 applies to the subtransition kernels $\{Q_k\}$,

$$\mathbb{E}^0[f(X(\tau))] = C(f),$$

where $C(f)$ is the same for all strong Markov, natural solutions of the constrained martingale problem for (A, D, B, Ξ). The same argument works for the hitting distribution of each E_n, by applying Theorem 2.4 to the subtransition kernels $\{Q_k\}_{k\geq n+1}$.

Thus we are reduced to verifying the assumptions of Theorem 2.4: This is the main object of the next two subsections. □

3.4 Estimates on hitting times

In this subsection we verify, for δ^* small enough and for an arbitrary $n \geq 0$, assumption (i) of Theorem 2.4 for the subtransition functions $\{Q_{n+k}\}$ defined by (3.18), (3.13) and (3.12), where X is a strong Markov, natural solution of the constrained martingale problem for (A, D, B, Ξ). Note that the subtransition function Q_{n+k} is the same for any natural solution, because it depends only on the path until the origin is hit. In this context, assumption (i) can be reformulated as: There exists $c_0 > 0$ such that

$$\inf_{x,y\in E_{n+k}} \frac{\mathbb{P}^x(\tau^n < \vartheta)}{\mathbb{P}^y(\tau^n < \vartheta)} \geq c_0, \quad \forall k \geq 1.$$

or, more generally, there exists $c_0 > 0$ such that, for $\delta \leq \delta^*$,

$$\inf_{0<|x|=|y|<\delta} \frac{\mathbb{P}^x(\tau^\delta < \vartheta)}{\mathbb{P}^y(\tau^\delta < \vartheta)} \geq c_0, \tag{3.19}$$

where τ^δ is defined by (3.8).

The proof of (3.19) is based on estimating $\mathbb{P}^x(\tau^\delta < \vartheta)$, both from above and from below, by means of suitable auxiliary functions (Lemmas 3.19 and 3.25). These auxiliary functions are constructed by elaborating on some functions introduced by Varadhan and Williams (1985) and Kwon and Williams (1991) in the study of the reflecting Brownian motion in $\overline{\mathcal{K}}$ with direction of reflection $\bar{g}$ (see Appendices A1 and A2).

Lemma 3.25 *There exists $\delta^* > 0$ such that:*

(i) For $\alpha^* \leq 0$, there exists a function $V \in \mathcal{C}^2(\overline{D} - \{0\})$ such that

$$\lim_{x\in\overline{D},\, x\to 0} V(x) = \infty \tag{3.20}$$

$$\nabla V(x) \cdot g(x) \leq 0, \quad \text{for } x \in \big(\partial D - \{0\}\big) \cap \overline{B_{\delta^*}(0)} \tag{3.21}$$

$$AV(x) \leq 0, \quad \text{for } x \in \big(\overline{D} - \{0\}\big) \cap \overline{B_{\delta^*}(0)}. \tag{3.22}$$

(ii) *For $0<\alpha^*<1$, there exist two functions $V_1, V_2 \in \mathcal{C}^2(\overline{D}-\{0\})$ such that*

$$V_1(x)>0, \quad V_2(x)>0, \quad \text{for } x \in \big(\overline{D}-\{0\}\big)\cap\overline{B_{\delta^*}(0)},$$

$$\lim_{x\in\overline{D},\, x\to 0} V_1(x) = \lim_{x\in\overline{D},\, x\to 0} V_2(x) = 0, \tag{3.23}$$

$$\inf_{0<\delta\le\delta^*} \frac{\inf_{|x|=\delta} V_1(x)}{\sup_{|x|=\delta} V_2(x)} > 0, \quad \inf_{0<\delta\le\delta^*} \frac{\inf_{|x|=\delta} V_2(x)}{\sup_{|x|=\delta} V_1(x)}, > 0$$

$$\nabla V_1(x)\cdot g(x) \ge 0, \quad \nabla V_2(x)\cdot g(x)\le 0, \quad \text{for } x\in\big(\partial D-\{0\}\big)\cap\overline{B_{\delta^*}(0)} \tag{3.24}$$

$$AV_1(x)\ge 0, \quad AV_2(x)\le 0, \quad \text{for } x\in\big(\overline{D}-\{0\}\big)\cap\overline{B_{\delta^*}(0)}. \tag{3.25}$$

Proof. See Appendix A2. □

Lemma 3.26 *Assume Conditions 3.1, 3.3 and 3.6. For a natural solution, X, of the constrained martingale problem for (A, D, B, Ξ), let*

$$\vartheta := \inf\{t\ge 0 : X(t)=0\}, \quad \tau^\delta := \inf\{t\ge 0 : |X(t)|=\delta\}, \quad \delta>0.$$

There exists δ^*, $0<\delta^*\le\bar{\delta}$, such that:

(i) *For $\alpha^*\le 0$, for each $x\in\overline{D}$, $0<|x|<\delta\le\delta^*$,*

$$\mathbb{P}^x\big(\tau^\delta<\vartheta\big)=1.$$

(ii) *For $0<\alpha^*<1$, there exists a positive constant c_0 such that, for $x, y\in\overline{D}$, $0<|x|=|y|<\delta\le\delta^*$, $\mathbb{P}^y(\tau^\delta<\vartheta)>0$ and*

$$\frac{\mathbb{P}^x(\tau^\delta<\vartheta)}{\mathbb{P}^y(\tau^\delta<\vartheta)}\ge c_0.$$

Proof. Let δ^*, V, V_1, V_2 be as in Lemma 3.25.

If $\alpha^*\le 0$, by applying Ito's formula to the function V, we have, for $\delta\le\delta^*$, for every fixed $x\in\big(\overline{D}-\{0\}\big)\cap B_\delta(0)$ and $\epsilon<|x|$,

$$\mathbb{E}^x[V(X(\tau^\epsilon\wedge\tau^\delta)]\le V(x),$$

which yields, V being nonnegative for $x\in\big(\overline{D}-\{0\}\big)\cap\overline{B_{\delta^*}(0)}$,

$$\inf_{|y|=\epsilon} V(y)\mathbb{P}^x\big(\tau^\epsilon<\tau^\delta\big)\le V(x),$$

and hence, by letting $\epsilon\to 0$,

$$\mathbb{P}^x\big(\vartheta<\tau^\delta\big)=0.$$

If $0<\alpha^*<1$, by applying Ito's formula to V_1, we obtain, for $x\in\big(\overline{D}-\{0\}\big)\cap B_\delta(0)$ and $\epsilon<|x|$,

$$\mathbb{E}^x[V_1(X(\tau^\delta\wedge\tau^\epsilon))]\ge V_1(x),$$

which yields

$$\sup_{|u|=\delta} V_1(u)\,\mathbb{P}^x\big(\tau^\delta < \tau^\epsilon\big) + \sup_{|u|=\epsilon} V_1(u)\mathbb{P}^x\big(\tau^\epsilon < \tau^\delta\big) \geq V_1(x),$$

and hence, by letting $\epsilon \to 0$,

$$\sup_{|u|=\delta} V_1(u)\,\mathbb{P}^x\big(\tau^\delta < \vartheta\big) \geq V_1(x). \tag{3.26}$$

Analogously, by applying Ito's formula to V_2 we get,

$$\inf_{|u|=\delta} V_2(u)\,\mathbb{P}^x\big(\tau^\delta < \tau^\epsilon\big) + \inf_{|u|=\epsilon} V_2(u)\,\mathbb{P}^x\big(\tau^\epsilon < \tau^\delta\big) \leq V_2(x),$$

and hence, by letting $\epsilon \to 0$,

$$\inf_{|u|=\delta} V_2(u)\,\mathbb{P}^x\big(\tau^\delta < \vartheta\big) \leq V_2(x). \tag{3.27}$$

Combining (3.26) and (3.27), we get, for $x, y \in \big(\overline{D} - \{0\}\big) \cap B_\delta(0)$ with $|x| = |y|$,

$$\frac{\mathbb{P}^x\big(\tau^\delta < \vartheta\big)}{\mathbb{P}^y\big(\tau^\delta < \vartheta\big)} \geq \frac{V_1(x)}{V_2(y)}\,\frac{\inf_{|u|=\delta} V_2(u)}{\sup_{|u|=\delta} V_1(u)} \geq \inf_{0<\delta\leq\delta^*} \frac{\inf_{|u|=\delta} V_1(u)}{\sup_{|u|=\delta} V_2(u)} \inf_{0<\delta\leq\delta^*} \frac{\inf_{|u|=\delta} V_2(u)}{\sup_{|u|=\delta} V_1(u)} > 0.$$

□

3.5 Estimates on hitting distributions

In this subsection we verify assumption (ii) of Theorem 2.4 by a scaling argument and a coupling argument similar to those of Costantini and Kurtz (2018). Assumption (ii) follows essentially from the fact that, for any x, $\widetilde{x} \in E_n$, we can construct, on the same probability space, two natural solutions of the constrained martingale problem for (A, D, B, Ξ), starting at x and $\widetilde{x}$, such that the probability that they hit E_{n-1} before the origin and that they couple before hitting E_{n-1} (i.e. that their paths agree, up to a time shift, for some time before they hit E_{n-1}) is larger than some $\epsilon_0 > 0$ independent of x and $\widetilde{x}$ and of n (Lemma 3.29). The construction is based on a result of Costantini and Kurtz (2018) and on a uniform lower bound on the probability that a natural solution of the constrained martingale problem for (A, D, B, Ξ) starting on E_n hits the intermediate layer $\{x \in \overline{D};\ |x| = 2^{-2n+1}\delta^*\}$ in the open set $\mathcal{O}^n := \{x \in D : 2^{2n-1}x \in \mathcal{O}\}$, where $\mathcal{O}$ is arbitrary. In turn this uniform lower bound is proved by the support theorem of Kwon and Williams (1991) and by showing that, for any natural solution of the constrained martingale problem for (A, D, B, Ξ), X, the rescaled process $2^{2n}\, X(2^{-4n}\cdot)$ converges to the reflecting Brownian motion in $\overline{\mathcal{K}}$ with direction of reflection $\bar{g}$ (Lemma 3.28). Existence and uniqueness of the reflecting Brownian motion in $\overline{\mathcal{K}}$ with direction of reflection $\bar{g}$ has been proved in Varadhan and Williams (1985), Williams (1985) and Kwon and Williams (1991), assuming only Conditions 3.1 (i) and (ii) and Conditions 3.3 (i) and (ii). We show in Appendix A1 (Theorem A1.9) that, if Conditions 3.3 (iii) and (iv) are verified, the reflecting Brownian motion is the unique natural solution of the constrained martingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, B, \bar{\Xi})$, where

$$\bar{\Xi} := \{(x, u) \in \partial\mathcal{K} \times S^{d-1} : u \in \bar{G}(x)\}, \quad \bar{G}(x) := \begin{cases} \{\eta\bar{g}(x),\ \eta \geq 0\}, & x \in \partial\mathcal{K} - \{0\}, \\ G(0), & x = 0, \end{cases}$$

(in particular the reflecting Brownian motion is a semimartingale). In this subsection, in order to distinguish the reflecting Brownian motion in $\overline{\mathcal{K}}$ with direction of reflection $\bar{g}$ from a solution of the constrained martingale problem for (A, D, B, Ξ), we will denote the former by $\bar{X}$.

Recall that we are assuming Conditions 3.1, 3.3 and 3.6 throughout this section.

Lemma 3.27 *For $\delta > 0$, let*

$$\bar{\tau}_\delta := \inf\{t \geq 0 : \, |\bar{X}(t)| \geq \delta\}, \quad \bar{\tau}_\delta^+ := \inf\{t \geq 0 : \, |\bar{X}(t)| > \delta\}.$$

Then, for every $\bar{x} \in \overline{\mathcal{K}} \cap \overline{B_\delta(0)}$,

$$\mathbb{P}^{\bar{x}}\big\{\bar{\tau}_\delta = \bar{\tau}_\delta^+\big\} = 1.$$

Proof. For $\bar{x} \in \partial B_\delta(0) \cap \mathcal{K}$, that is $|\bar{x}| = \delta$ and $\bar{x}$ in the interior of the cone, then, by standard properties of Brownian motion,

$$\mathbb{P}^{\bar{x}}(\bar{\tau}_\delta^+ = 0) = 1. \tag{3.28}$$

For $\bar{x} \in \partial B_\delta(0) \cap \partial\mathcal{K}$, note that, for all r,

$$|\partial B_r(x) \cap (\overline{B_\delta})^c \cap \overline{\mathcal{K}}| \geq \frac{1}{2}|\partial B_r(x) \cap \overline{\mathcal{K}}|.$$

and let

$$\bar{\theta}_r := \inf\{t \geq 0 : \, \bar{X}(t) \notin B_r(\bar{x})\}.$$

Then, by Lemmas 3.3 and 2.3 of Kwon and Williams (1991), for $r < |\bar{x}|$,

$$\mathbb{P}^{\bar{x}}(\bar{\tau}_\delta^+ \leq \bar{\theta}_r) \geq \mathbb{P}^{\bar{x}}(\bar{X}(\bar{\theta}_r) \in \partial B_r(\bar{x}) \cap (\overline{B_\delta})^c \cap \overline{\mathcal{K}}) \geq \kappa > 0,$$

where, for r sufficiently small, κ is independent of r. Since $\bar{\theta}_r \to_{r\to 0} 0$ a.s., this implies

$$\mathbb{P}^{\bar{x}}(\bar{\tau}_\delta^+ = 0) \geq \kappa,$$

and hence, by the strong Markov property of $\bar{X}$ and the 0-1 law (see Proposition 7.7 and the proof of Theorem 7.17 of Karatzas and Shreve (1991)), (3.28) holds for $\bar{x} \in \partial B_\delta(0) \cap \partial\mathcal{K}$ as well.

Then the assertion follows by the strong Markov property. □

Let X be a natural solution of the constrained martingale problem for (A, D, B, Ξ). Define,

$$\tau^{(n-1)\prime} := \inf\{t \geq 0 : |X(t)| = 2^{-2n+1}\delta^*\}, \tag{3.29}$$

and note that $\tau^{(n-1)\prime}$ is the hitting time of the surface "halfway" between E_n and E_{n-1}. Consistently with (3.14), we denote

$$\tau^{0\prime} = \tau'.$$

Recall that we have defined $\vartheta := \inf\{t \geq 0 : X(t) = 0\}$

Lemma 3.28 *For any sequence $\{x^n\} \subseteq \overline{D}$ such that $\{2^{2n}x^n\}$ converges to some $\bar{x} \in \overline{\mathcal{K}} - \{0\}$, let X^n be a natural solution of the constrained martingale problem for (A, D, B, Ξ) starting at x^n and $\bar{X}$ be the reflecting Brownian motion in $\overline{\mathcal{K}}$ with direction of reflection $\bar{g}$, starting at $\bar{x}$. Then*

$$2^{2n} X^{x^n}(2^{-4n}\cdot) \xrightarrow{\mathcal{L}} \bar{X}^{\bar{x}}(\cdot). \tag{3.30}$$

In particular, for any open set $\mathcal{O}$ such that $\mathcal{O} \cap \mathcal{K} \cap \partial B_{\delta^*/2}(0) \neq \emptyset$, there exists $\eta_0 = \eta_0(\mathcal{O}) > 0$ such that, for $|x^n| = 2^{-2n-2}\delta^*$, hence $|\bar{x}| = \delta^*/4$, and $\mathcal{O}^n := \{x : 2^{2n+1}x \in \mathcal{O}\}$, it holds

$$\liminf_{n\to\infty} \mathbb{P}^{x^n}(\tau^{(n-1)\prime} < \vartheta, \;\; X(\tau^{(n-1)\prime}) \in \mathcal{O}_n) \geq \eta_0., \tag{3.31}$$

Proof. The convergence in (3.30) follows from compactness arguments such as, for instance, those used in Lemma 2.8 of Costantini and Kurtz (2019) and from the fact that $\bar{X}$ is the unique solution of the constrained martingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, B, \bar{\Xi})$ (Theorem A1.9).

As for (3.31), (3.30) together with Lemma 3.27 yield

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbb{P}^{x^n}(\tau^{(n-1)\prime} < \vartheta, \;\; X(\tau^{(n-1)\prime}) \in \mathcal{O}_n) \\
&\geq \mathbb{P}^{\bar{x}}\{\bar{\tau}' < \bar{\vartheta}, \, \bar{X}(\bar{\tau}') \in \mathcal{O}\} \geq \inf_{x \in \overline{\mathcal{K}}, \, |x| = \delta^*/4} \mathbb{P}^{x}\{\bar{\tau}' < \bar{\vartheta}, \, \bar{X}(\bar{\tau}') \in \mathcal{O}\}.
\end{aligned}$$

where $\bar{\tau}' := \inf\{t : |\bar{X}(t)| \geq 2^{-1}\delta^*\}$ and $\bar{\vartheta} = \inf\{t : \bar{X}(t) = 0\}$. (3.31) then follows from the support theorem of Kwon and Williams (1991) (Theorem 3.1), the Feller property and Lemma 3.27. $\square$

Lemma 3.29 *Let $\{D_n\}$ be defined by* (3.12) *and $\{Q_n\}$ be given by* (3.18)-(3.13). *With the notation of Lemma 2.3 and Theorem 2.4, for $x, \widetilde{x} \in \partial D_n$, let*

$$\epsilon_n(x, \widetilde{x}) = \int \big(\widetilde{f}_n(x, y) \wedge f_n(\widetilde{x}, y)\big)\big(Q_n(x, dy) + Q_n(\widetilde{x}, dy)\big), \quad n \geq 1,$$

where

$$\widetilde{f}_n(x, \cdot) = \frac{dQ_n(x, \cdot)}{d(Q_n(x, \cdot) + Q_n(\widetilde{x}, \cdot))}, \quad f_n(\widetilde{x}, \cdot) = \frac{dQ_n(\widetilde{x}, \cdot)}{d(Q_n(x, \cdot) + Q_n(\widetilde{x}, \cdot))}.$$

Then there exists $\epsilon_0 > 0$ such that

$$\inf_{n \geq 1} \inf_{x, \widetilde{x} \in \partial D_n} \epsilon_n(x, \widetilde{x}) \geq \epsilon_0.$$

Proof. By Lemma 3.28 and Lemma 5.3 in Costantini and Kurtz (2018), we can construct, on the same probability space, two natural solutions, X and $\widetilde{X}$, of the constrained martingale problem for (A, D, B, Ξ), starting at x and $\widetilde{x}$ respectively, such that, denoting by $\mathcal{E}$ the event

$$\mathcal{E} := \{\exists t, \widetilde{t}, \, 0 \leq t < \tau^{n-1} \wedge \vartheta, \, \widetilde{t} < \widetilde{\tau}^{n-1} \wedge \widetilde{\vartheta} : \; X(t+s) = \widetilde{X}(\widetilde{t}+s), \, 0 \leq s \leq (\tau^{n-1} \wedge \vartheta) - t\},$$

it holds

$$\mathbb{P}(\{\tau^{n-1} < \vartheta\} \cap \mathcal{E}) = \mathbb{P}(\{\widetilde{\tau}^{n-1} < \widetilde{\vartheta}\} \cap \mathcal{E}) \geq \epsilon_0,$$

for some positive constant ϵ_0 independent of n, x and $\widetilde{x}$. This implies

$$\|Q_n(x,\cdot)-Q_n(\widetilde{x},\cdot)\| \le Q_n(x,E_{n-1})\wedge Q_n(\widetilde{x},E_{n-1})-\epsilon_0.$$

On the other hand, we have

$$\begin{aligned}\|Q_n(x,\cdot)-Q_n(\widetilde{x},\cdot)\| &= \int \big(f_n(x,y)\vee \widetilde{f}_n(\widetilde{x},y)\big)\big(Q_n(x,dy)+Q_n(\widetilde{x},dy)\big)-\epsilon_n(x,\widetilde{x}) \\ &\ge Q_n(x,E_{n-1})\wedge Q_n(\widetilde{x},E_{n-1})-\epsilon_n(x,\widetilde{x}),\end{aligned}$$

and the assertion follows by combining this inequality with the previous one. □

A1 Results on the cone

Let $\mathcal{K}$ be the cone in Condition 3.1 and $\bar{g}$ be the vector field in Condition 3.3. The reflecting Brownian motion in $\overline{\mathcal{K}}$ with direction of reflection $\bar{g}$ has been studied by Varadhan and Williams (1985) and Williams (1985), for $d=2$, and by Kwon and Williams (1991) for $d\ge 3$, without assuming Conditions 3.3 (iii) and (iv). We summarize below the main results of Varadhan and Williams (1985) and Kwon and Williams (1991).

If $\bar{g}$ satisfies Conditions 3.3 (iii) and (iv), a modification of Theorem 3.22 (Theorem A1.9 below) yields that the reflecting Brownian in $\overline{\mathcal{K}}$ with direction of reflection $\bar{g}$ is a semimartingale. In dimension $d=2$, Williams (1985) proves that this is equivalent to the fact that the parameter α^* (defined in $(A1.3)$ below) is strictly less than 1. In dimension $d\ge 3$, the issue of when the reflecting Brownian motion is a semimartingale is not discussed in Kwon and Williams (1991). We prove here one of the two implications, namely that the fact that the reflecting Brownian motion is a semimartingale implies that the parameter α^* (defined by Kwon and Williams (1991) as in Theorem A1.4 is strictly less than 1 (Theorem A1.10). Beyond the intrinsec interest, this allows us to approximate the domain D by the cone $\mathcal{K}$ in some of the estimates we need to prove uniqueness. (see the proof of Lemma **??**).

Let $\mathcal{K}$ be a cone as in Conditions 3.1 (i) and (ii), $\bar{g}$ be a vector field as in Conditions 3.3 (i) and (ii). In Varadhan and Williams (1985), Williams (1985) and Kwon and Williams (1991), the reflecting Brownian motion in $\overline{\mathcal{K}}$ with direction of reflection $\bar{g}$ is viewed as a solution to the following submartingale problem.

Definition A1.1 *A stochastic process X with paths in $\mathcal{C}_{\overline{\mathcal{K}}}[0,\infty)$ is a solution of the* submartingale problem *for $(\frac{1}{2}\Delta,\mathcal{K},\bar{g}\cdot\nabla,\partial\mathcal{K})$, if there exists a filtration $\{\mathcal{F}_t\}$, on the space on which X is defined, such that X is $\{\mathcal{F}_t\}$-adapted and*

$$f(X(t))-f(X(0))-\frac{1}{2}\int_0^t \Delta f(X(s))\,ds$$

is an $\{\mathcal{F}_t\}$-submartingale for all $f\in\mathcal{C}_b^2(\overline{\mathcal{K}})$ such that f is constant in a neighborhood of the origin and

$$\bar{g}\cdot\nabla f\ge 0 \quad \text{on } \partial\mathcal{K}-\{0\}.$$

The solution to the submartingale problem for $(\frac{1}{2}\Delta,\overline{\mathcal{K}},\bar{g}\cdot\nabla,\partial\mathcal{K})$ is unique if any two solutions have the same distribution.

A solution X is said to spend zero time at the origin if

$$\mathbb{E}\Big[\int_0^\infty \mathbf{1}_{\{0\}}(X(s))ds\Big] = 0$$

For $d = 2$, in polar coordinates, we can suppose, without loss of generality,

$$\mathcal{K} = \{(r, z) :\ r > 0,\ 0 < z < \zeta\}, \quad 0 < \zeta < 2\pi, \tag{A1.1}$$

Let $\partial_1\mathcal{K} := \{(r, z) :\ r > 0,\ z = 0\}$, $\partial_2\mathcal{K} := \{(r, o) :\ r > 0,\ z = \zeta\}$ and denote by n^1 and n^2 the unit inward normal vectors on $\partial_1\mathcal{K}$ and $\partial_2\mathcal{K}$. Conditions 3.3 (i) and (ii) reduce simply to

$$\bar{g}(x) := \begin{cases} \bar{g}^1, & \text{for } x \in \partial_1\mathcal{K}, \\ \bar{g}^2, & \text{for } x \in \partial_2\mathcal{K}. \end{cases} \tag{A1.2}$$

Theorem A1.2 (Varadhan and Williams (1985))
Let $d = 2$, and let $\mathcal{K}$ and $\bar{g}$ be as in (A1.1) and (A1.2). Let ζ_1 and ζ_2 denote the angles between $\bar{g}^1$ and n^1, and between $\bar{g}^2$ and n^2, respectively, taken to be positive if $\bar{g}^1$ ($\bar{g}^2$) points towards the origin. Set

$$\alpha^* := \frac{\zeta_1 + \zeta_2}{\zeta}, \tag{A1.3}$$

and

$$\begin{aligned} \psi_{\alpha^*}(z) &:= \cos(\alpha^* z - \zeta_1), && \text{if } \alpha^* \neq 0, \\ \psi^0(z) &:= -z\mathrm{tg}\ \zeta_1, && \text{if } \alpha^* = 0. \end{aligned} \tag{A1.4}$$

Then the function

$$\Psi(r, z) := \begin{cases} r^{\alpha^*}\psi_{\alpha^*}(z), & \text{if } \alpha^* \neq 0, \\ -\ln r\ +\ \psi^0(z), & \text{if } \alpha^* = 0, \end{cases} \tag{A1.5}$$

satisfies

$$\begin{aligned} \Delta\Psi = 0 \qquad & \text{in } \mathcal{K}, \\ \bar{g}\cdot\nabla\Psi = 0 \qquad & \text{on } \partial\mathcal{K} - \{0\}. \end{aligned} \tag{A1.6}$$

Theorem A1.3 (Varadhan and Williams (1985), Williams (1985))
Let $d = 2$, and let $\mathcal{K}$, $\bar{g}^1$, $\bar{g}^2$, α^* be as in Theorem A1.2. For $\alpha^* < 2$, for each $x \in \overline{\mathcal{K}}$, there exists one and only one solution to the submartingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, \bar{g}\cdot\nabla, \partial\mathcal{K})$ starting at x that spends zero time at the origin and it is a strong Markov process and a Feller process. This solution is a semimartingale if and only if $\alpha^* < 1$. For $\alpha^* \geq 2$, for each $x \in \overline{\mathcal{K}}$, there exists one and only one solution to the submartingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, \bar{g}\cdot\nabla, \partial\mathcal{K})$ starting at x, and it is absorbed at the origin after the first time it hits it.

Now let $d \geq 3$. Recall that n^r denotes the unit radial vector, i.e. $n^r(x) := \frac{x}{|x|}$, and $n^{\mathcal{K}}(x)$ denotes the unit inward normal to $\overline{\mathcal{K}}$ at $x \neq 0$.

Theorem A1.4 (Kwon and Williams (1991))
Let $d \geq 3$, $\mathcal{K}$ be a cone as in Conditions 3.1 (i) and (ii), $\bar{g}$ be a vector field as in Conditions 3.3 (i) and (ii). For each $\alpha \in \mathbb{R}$ there exist $\lambda(\alpha) \in \mathbb{R}$ and $\psi_\alpha \in \mathcal{C}^2(\overline{\mathcal{S}})$ such that

$$\begin{aligned} \lambda(\alpha)\,\psi_\alpha + \Delta_{S^{d-1}}\psi_\alpha = 0 \qquad & \text{in } \mathcal{S}, \\ \alpha\bar{g}_r\psi_\alpha + \bar{g}_T\cdot\nabla_{S^{d-1}}\psi_\alpha = 0 \qquad & \text{on } \partial\mathcal{S}, \end{aligned} \tag{A1.7}$$

where $\bar{g}_r n^r$ and $\bar{g}_T$ are the radial component and the component tangential to S^{d-1} of $\bar{g}$. ψ_α is strictly positive. λ and $\alpha \mapsto \psi_\alpha \in \mathcal{C}^2(\overline{\mathcal{S}})$ are analytic functions. λ is concave, $\lambda(0) = 0$ and

$$\lambda'(0) = -\int_{\partial\mathcal{S}} \frac{1}{\bar{g} \cdot n^{\mathcal{K}}} \bar{g}_r \psi^*,$$

where ψ^* is the unique solution of

$$\begin{aligned} \Delta_{S^{d-1}} \psi^* = 0 \qquad &\text{in } \mathcal{S}, \\ n^{\mathcal{K}} \cdot \nabla_{S^{d-1}} \psi^* - \operatorname{div}_{\partial\mathcal{S}}\big(\psi^*(\tfrac{1}{\bar{g}\cdot n^{\mathcal{K}}} \bar{g}_T - n^{\mathcal{K}})\big) = 0 \qquad &\text{on } \partial\mathcal{S}, \end{aligned} \tag{A1.8}$$

such that ψ^* is strictly positive and $\int_{\mathcal{S}} \psi^* = 1$.

If $\lambda'(0) \neq d-2$, there exists a unique $\alpha^* \neq 0$ such that

$$\lambda(\alpha^*) = \alpha^*(\alpha^* + d - 2),$$

and the function Ψ defined as in (A1.5) for $\alpha^* \neq 0$ satisfies (A1.6). $\alpha^* > 0$ if $\lambda'(0) > d-2$, $\alpha^* < 0$ if $\lambda'(0) < d-2$.

If $\lambda'(0) = d-2$, there exists a solution to

$$\begin{aligned} -(d-2) + \Delta_{S^{d-1}} \psi^0 = 0, \qquad &\text{in } \mathcal{S}, \\ -\bar{g}_r + \bar{g}_T \cdot \nabla_{S^{d-1}} \psi^0 = 0, \qquad &\text{on } \partial\mathcal{S}, \end{aligned}$$

and the function Ψ defined as in (A1.5) for $\alpha^* = 0$ satisfies (A1.6). In this case, we set $\alpha^* := 0$.

Theorem A1.5 (Kwon and Williams (1991))
Let $d \geq 3$ and let $\mathcal{K}$ and $\bar{g}$ be as in Theorem A1.4. For $\alpha^* < 2$, for each $x \in \overline{\mathcal{K}}$, there exists a unique solution to the submartingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, \bar{g} \cdot \nabla, \partial\mathcal{K})$, starting at x, that spends zero time at the origin and it is a strong Markov process and a Feller process. For $\alpha^* \geq 2$, for each $x \in \overline{\mathcal{K}}$, there exists a unique solution to the submartingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, \bar{g} \cdot \nabla, \partial\mathcal{K})$ starting at x, and it is absorbed at the origin after the first time it hits it.

Remark A1.6 *In the case $\alpha^* = 0$, both for $d = 2$ and $d \geq 3$, the function used in Kwon and Williams (1991) is actually $-\Psi$, but we prefer to have $\Psi(r, z) \to_{r \to 0} \infty$.*

Remark A1.7 *For $d = 2$, $\psi_{\alpha^*}, \psi^0 \in \mathcal{C}^\infty(\overline{\mathcal{S}})$. For $d \geq 3$, a careful inspection of the proofs of Kwon and Williams (1991) shows that $\psi_{\alpha^*}, \psi^0 \in \mathcal{C}^{2+\beta}(\overline{\mathcal{S}})$ for every $0 < \beta < 1$ (see Theorem 6.31 of Gilbarg and Trudinger (1983) and the Remark following it).*

The function defined in (2.7) of Kwon and Williams (1991):

$$\Phi(x) := \begin{cases} \Psi(x)^{-1}, & \text{if } \alpha^* < 0, \\ e^{-\Psi(x)}, & \text{if } \alpha^* = 0, \\ \Psi(x), & \text{if } \alpha^* > 0, \end{cases} \tag{A1.9}$$

gives a way of measuring the distance from the origin and satisfies $\bar{g} \cdot \nabla\Phi = 0$ on $\partial\mathcal{K} - \{0\}$. It will be used both to localize and to construct auxiliary functions (see Appendix A2).

Let $\bar{g}$ be a vector field as in Conditions 3.3 (i) and (ii) and let $G(0)$ be the closed, convex cone generated by $\{\bar{g}(z),\, z \in \partial\mathcal{S}\}$. Set

$$\mathcal{D}(\Delta) := \mathcal{C}_b^2(\overline{\mathcal{K}}),$$

$$\bar{G}(x) := \begin{cases} \{\eta\bar{g}(x),\, \eta \geq 0\}, & x \in \partial\mathcal{K} - \{0\}, \\ G(0), & x = 0, \end{cases}$$

$$\bar{\Xi} := \{(x,u) \in \partial\mathcal{K} \times U : u \in \bar{G}(x)\}, \quad U := S^{d-1}, \tag{A1.10}$$

$$Bf(x,u) := \nabla f(x) \cdot u, \quad \mathcal{D}(B) := \mathcal{C}_b^2(\overline{\mathcal{K}})$$

$\mathcal{K}$ is unbounded, but the definitions of constrained martingale problem, controlled martingale problem and natural solution of the constrained martingale problem carry over to $(\frac{1}{2}\Delta, \mathcal{K}, B, \bar{\Xi})$ without any modification.

Lemma A1.8 *There exists a function $F \in \mathcal{C}_b^2(\overline{\mathcal{K}})$ such that*

$$\inf_{x \in \partial\mathcal{K}-\{0\}} \nabla F(x) \cdot \bar{g}(x) := c_F > 0.$$

Proof. See Appendix A2. $\square$

Theorem A1.9 *Let $\mathcal{K}$ and $\bar{g}$ be as in Theorem A1.2, for $d = 2$, and as in Theorem A1.4, for $d \geq 3$, and, in addition, assume that $\bar{g}$ satisfies Conditions 3.3 (iii) and (iv).*

Then, for each $\nu \in \mathcal{P}(\bar{\mathcal{K}})$, there exists one and only one natural solution, X, to the constrained martingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, B, \bar{\Xi})$ with initial distribution ν and it is the unique solution of the submartingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, \bar{g}\cdot\nabla, \partial\mathcal{K})$ that spends zero time at the origin. X is a semimartingale. The associated random measure, Λ, satisfies $\mathbb{E}[\Lambda([0,t] \times \bar{\Xi})] < \infty$ for all $t \geq 0$, and (3.4) is a martingale.

Proof. Let $\{\delta_k\}$ be a strictly decreasing sequence of positive numbers converging to zero, with $\delta_1 < 1$, and let $\{D^k\}$ be a sequence of domains with $\mathcal{C}^1$ boundary such that $D^k \subset D^{k+1} \subset \mathcal{K}$, $\overline{D^k} \cap \left(B_{\delta_k}(0)\right)^c = \overline{\mathcal{K}} \cap \left(B_{\delta_k}(0)\right)^c$ and $\overline{D^k} \cap B_{\delta_k}(0) \subset D^{k+1}$. Also let $g^k : \mathbb{R}^d - \{0\} \to \mathbb{R}^d$, $k \in \mathbb{N}$, be a locally Lipschitz vector field, of unit length on ∂D^k, such that $g^k(x) = \bar{g}(x)$ for $x \in \partial\mathcal{K} \cap \left(B_{\delta_k}(0)\right)^c$ and that, denoting by $n^k(x)$ the unit, inward normal at $x \in \partial D^k$, it holds $\inf_{x\in\partial D^k} g^k(x) \cdot n^k(x) > 0$.

For each k, consider a sequence of bounded domains $\{D^{k,N}\}$, $N \in \mathbb{N}$, with $\mathcal{C}^1$ boundary, such that $D^{k,N} \subset D^{k,N+1} \subset D^k$, $\overline{D^{k,N}} \cap \overline{B_N(0)} = \overline{D^k} \cap \overline{B_N(0)}$ and $\overline{D^{k,N}} \cap \left(\overline{B_N(0)}\right)^c \subset D^{k,N+1}$. Also let $g^{k,N}$ be a locally Lipschitz vector field, of unit length on $\partial D^{k,N}$, such that $g^{k,N}(x) = g^k(x)$ for $x \in \partial D^k \cap \overline{B_N(0)}$ and that, denoting by $n^{k,N}(x)$ the unit, inward normal at $x \in \partial D^{k,N}$, it holds $\inf_{x\in\partial D^k} g^{k,N}(x) \cdot n^{k,N}(x) > 0$.

Let ξ_0 be a random variable with compact support $\mathrm{supp}(\xi_0) \subset \overline{\mathcal{K}} - \{0\}$ and, for k and N large enough that $\mathrm{supp}(\xi_0) \subset \overline{D^{k,N}}$, let $\xi^{k,N}$ be the (strong) solution of (3.1) in $\overline{D^{k,N}}$

with direction of reflection $g^{k,N}$ and initial condition ξ_0, and let $l^{k,N}$ be the corresponding nondecreasing process. Define

$$\Theta^{k,N} := \inf\{t \geq 0 : \, |\xi^{k,N}(t)| \geq N\}.$$

Let $\varphi \in \mathcal{C}^2(\overline{\mathcal{K}})$ be defined by:

$$\varphi(x) := \chi(\Phi(x)), \tag{A1.11}$$

where Φ is defined in $(A1.9)$ and $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth, nondecreasing function such that $\chi(u) = 0$, for $u \leq \sup_{|x|\leq\delta} \Phi(x)$ and $\chi(u) = u$, for $u \geq \inf_{|x|\geq\delta'} \Phi(x)$, for $1 \leq \delta < \delta'$ such that $0 < \sup_{|x|\leq\delta} \Phi(x) < \inf_{|x|\geq\delta'} \Phi(x)$. Then $\delta_k < \delta$ for all k and

$$\nabla\varphi(x) \cdot g^{k,N}(x) = 0, \text{ for } x \in \partial D^{k,N}, \, |x| \leq N,$$

$$\lim_{|x|\to\infty} \varphi(x) = \infty, \quad \Delta\varphi(x) \leq c\big(1 + \varphi(x)\big), \text{ for } x \in \mathcal{K}, \; |x| \leq N,$$

and, by applying Ito's formula to φ, we obtain

$$\lim_{N\to\infty} \sup_{k:\, \mathrm{supp}(\xi_0)\subset\overline{D^k}} \mathbb{P}\big(\Theta^{k,N} \leq t\big) = 0. \tag{A1.12}$$

From $(A1.12)$, by a standard procedure, we see that, for $\mathrm{supp}(\xi_0) \subset \overline{D_k}$, there is one and only one solution, ξ^k, to (3.1) in $\overline{D^k}$ with direction of reflection g^k and initial condition ξ_0, and it is defined for all times. Moreover, setting

$$\theta_k := \inf\{t \geq 0 : \, \xi^k(t) \in \partial D^k \cap B_{\delta_k}(0)\}, \quad \theta := \lim_{k\to\infty} \theta_k,$$

$(A1.12)$ yields that

$$\mathbb{P}\Big(\theta < \infty, \sup_{k:\, \mathrm{supp}(\xi_0)\subset\overline{D^k}} \sup_{t\leq\theta_k} |\xi^k(t)| = \infty\Big) = 0. \tag{A1.13}$$

Hence, by an analogous procedure, we can define a pair of stochastic processes ξ and l that satisfies (3.9) for $0 \leq t < \theta$, and almost every path of ξ such that $\theta < \infty$ is bounded. For each $N \in \mathbb{N}$, for each path such that $\theta < \infty$ and $\sup_{t<\infty} |\xi(t)| \leq N$, we can repeat the argument of Theorem 3.22 and obtain (3.10), so that (3.10) holds almost for every path such that $\theta < \infty$. Therefore the solution of (3.9) is well defined, up to θ included if θ is finite.

We can now proceed as in the proof of Theorem 3.22 and construct a sequence $\{(Y^n, \lambda_0^n, \Lambda_1^n)\}$ such that, for each $f \in \mathcal{C}_b^2(\overline{\mathcal{K}})$,

$$f(Y^n(t)) - f(Y^n(0)) - \frac{1}{2}\int_0^t \Delta f(Y^n(s))d\lambda_0^n(s) - \int_{[0,t]\times U} B^n f(Y^n(s), u)\Lambda_1^n(ds \times du)$$

is a martingale with respect to $\{\mathcal{F}_t^{Y^n,\lambda_0^n,\Lambda_1^n}\}$, where

$$B^n f(x, u) := u \cdot \nabla f(x)\mathbf{1}_{\partial\mathcal{K}-\{0\}}(x) + (\rho_n)^{-1}[f(x + \rho_n u) - f(x)]\mathbf{1}_{\{0\}}(x),$$

and $Y^n(0)$ is an arbitrary random variable with compact support in $\overline{\mathcal{K}} - \{0\}$. By employing again the function φ defined in $(A1.11)$, we can see that, if the laws of $Y^n(0)$ converge to

ν, $\{Y^n\}$ satisfies the compact containment condition. Then the same relative compactness arguments as in Theorem 3.22 apply and any limit point of $\{(Y^n, \lambda_0^n, \Lambda_1^n)\}$ will be a solution of the controlled martingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, B, \Xi)$ with initial distribution ν.

Lemma 3.1 in Costantini and Kurtz (2019) holds for non compact state spaces as well, provided that f and Af in its statement are bounded, and Lemma A1.8 ensures that its assumption is verified. Therefore, for every solution, $(Y, \lambda_0, \Lambda_1)$, of the controlled martingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, B, \overline{\Xi})$ and λ_0^{-1} defined by (3.7), it holds $\mathbb{E}\big[\lambda_0^{-1}(t)\big] < \infty$, which ensures that $Y \circ \lambda_0^{-1}$ is a solution of the constrained martingale problem and that (3.4) is a martingale. Since the function $f(x) := x_i$, $i = 1, ..., d$, can be approximated, uniformly over compact sets, by functions in $\mathcal{D}(\Delta) = \mathcal{D}(B)$, $Y \circ \lambda_0^{-1}$ is a semimartingale.

Moreover, it can be easily checked, in the same way as in Remark 3.12, that all solutions to the controlled martingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, B, \overline{\Xi})$ are continuous and that Remark 3.17 and Lemma 3.21 carry over to the present context. Therefore every natural solution of the constrained martingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, B, \overline{\Xi})$ is a solution of the submartingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, \bar{g} \cdot \nabla, \partial\mathcal{K})$ that spends zero time at the origin. □

Theorem A1.10 *Let $\mathcal{K}$ and $\bar{g}$ be as in Theorem A1.2, for $d = 2$, and as in Theorem A1.4, for $d \geq 3$, and, in addition, assume that $\bar{g}$ satisfies Conditions 3.3 (iii) and (iv). Let α^* be the parameter defined in Theorems A1.3 and A1.3. Then it holds*

$$\alpha^* < 1.$$

Proof. In dimension $d = 2$, the assertion follows immediately from Theorems A1.9 and A1.3. By adapting, in a nontrivial way, an argument of Williams (1985), we are able to prove that it holds in dimension $d \geq 3$ as well.

By Theorems A1.9 and A1.5, $\alpha^* < 2$. Suppose, by contradiction, that $1 \leq \alpha^* < 2$.

In the following, it is convenient to normalize $\bar{g}$ so that $\bar{g}(x) \cdot n^{\mathcal{K}}(x) = 1$, rather than $|\bar{g}(x)| = 1$. Of course this does not affect equation (A1.7) and Condition 3.3. It can be easily checked that, for $\epsilon > 0$ less than a threshold determined by the data of the problem, the vector

$$g^\epsilon(z) := \bar{g}(z) - \epsilon\, n^r, \quad z \in \partial\mathcal{S}, \tag{A1.14}$$

satisfies all points of Condition 3.3. Then, by Theorems A1.9 and A1.5, $\alpha^{\epsilon*}$, defined as in Theorem A1.4 with $\bar{g}$ replaced by g^ϵ, satisfies

$$\alpha^{\epsilon*} < 2. \tag{A1.15}$$

Let us show that

$$\alpha^{\epsilon*} > \alpha^*. \tag{A1.16}$$

For $\alpha > 0$, let $(\lambda(\alpha), \psi_\alpha)$ be as in Theorem A1.4 and $(\lambda^\epsilon(\alpha), \psi_\alpha^\epsilon)$ be the corresponding objects with $\bar{g}$ replaced by g^ϵ. Since $g_T^\epsilon = \bar{g}_T$, and $g_r^\epsilon = \bar{g}_r - \epsilon$, $(\lambda^\epsilon(\alpha), \psi_\alpha^\epsilon)$ satisfies

$$\begin{aligned} \lambda^\epsilon(\alpha)\, \psi_\alpha^\epsilon + \Delta_{S^{d-1}} \psi_\alpha^\epsilon = 0, \qquad & \text{in } \mathcal{S}, \\ \alpha(\bar{g}_r - \epsilon)\psi_\alpha^\epsilon + \bar{g}_T \cdot \nabla_{S^{d-1}} \psi_\alpha^\epsilon = 0, \qquad & \text{on } \partial\mathcal{S}. \end{aligned} \tag{A1.17}$$

Consider the function $\psi_\alpha(\psi^\epsilon_\alpha)^{-1}$. Straightforward computations show that $(A1.7)$ and $(A1.17)$ imply that, for $z \in \mathcal{S}$,

$$\Delta_{S^{d-1}}(\psi_\alpha(\psi^\epsilon_\alpha)^{-1})(z) = [\lambda^\epsilon(\alpha)-\lambda(\alpha)](\psi_\alpha(\psi^\epsilon_\alpha)^{-1})(z)-2\big((\psi^\epsilon_\alpha)^{-1}\nabla_{S^{d-1}}(\psi_\alpha(\psi^\epsilon_\alpha)^{-1})\cdot\nabla_{S^{d-1}}(\psi^\epsilon_\alpha)\big)(z),$$

and, for $z \in \partial\mathcal{S}$,

$$\big(\nabla_{S^{d-1}}(\psi_\alpha(\psi^\epsilon_\alpha)^{-1})\cdot \bar{g}_T\big)(z) = -\alpha\,\epsilon\,(\psi_\alpha(\psi^\epsilon_\alpha)^{-1})(z).$$

Let z^0 be a point of global minimum for $\psi_\alpha(\psi^\epsilon_\alpha)^{-1}$. If $z^0 \in \partial\mathcal{S}$, it must hold

$$\big(\nabla_{S^{d-1}}(\psi_\alpha(\psi^\epsilon_\alpha)^{-1})\cdot \bar{g}_T\big)(z^0) = \big(\nabla_{S^{d-1}}(\psi_\alpha(\psi^\epsilon_\alpha)^{-1})\cdot n^{\mathcal{K}}\big)(z^0) \geq 0,$$

while

$$-\alpha\,\epsilon\,(\psi_\alpha(\psi^\epsilon_\alpha)^{-1})(z^0) < 0, \qquad \forall\alpha > 0,$$

because ψ_α and $\widetilde{\psi}_\alpha$ are strictly positive. Therefore it must be $z^0 \in \mathcal{S}$ and

$$\nabla_{S^{d-1}}(\psi_\alpha(\psi^\epsilon_\alpha)^{-1})(z^0) = 0, \quad \Delta_{S^{d-1}}(\psi_\alpha(\psi^\epsilon_\alpha)^{-1})(z^0) > 0,$$

which yields

$$\lambda^\epsilon(\alpha) > \lambda(\alpha), \qquad \forall\alpha > 0. \tag{A1.18}$$

Then $\big(\lambda^\epsilon\big)'(0) \geq \lambda'(0) > d-2$, so that $\alpha^{\epsilon *} > 0$. Hence, taking into account that $\lambda(\alpha) - \alpha(\alpha+d-2)$ vanishes for $\alpha = 0$ and $\alpha = \alpha^*$ and is strictly concave, $(A1.18)$ gives $(A1.16)$.

The function Ψ^ϵ defined by $(A1.7)$ with ψ_α replaced by ψ^ϵ_α and α^* replaced by $\alpha^{\epsilon *}$ has the following properties:

$$\begin{gathered} \Delta\Psi^\epsilon(x) = 0, \quad \text{in } \mathcal{K}, \\ \epsilon c_1 \Psi(x)^{(\alpha^{\epsilon *}-1)/\alpha^*} \leq \big(\bar{g}\cdot\nabla\Psi^\epsilon\big)(x) \leq \epsilon c_2 \Psi(x)^{(\alpha^{\epsilon *}-1)/\alpha^*}, \quad \text{on } \partial\mathcal{K}, \\ c_1\Psi(x)^{\alpha^{\epsilon *}/\alpha^*} \leq \Psi^\epsilon(x) \leq c_2\Psi(x)^{\alpha^{\epsilon *}/\alpha^*}, \end{gathered} \tag{A1.19}$$

where the constants c_1 and c_2 can be taken independent of ϵ because the map $\alpha \to \psi_\alpha$ is continuous and $1 \leq \alpha^{\epsilon *} \leq 2$. Of course $(A1.19)$ still holds if we revert to the usual normalization of $\bar{g}$, $|\bar{g}| = 1$, as we will do for the rest of the proof.

The rest of the proof follows closely the proof of Theorem 5 of Williams (1985). Let X be the solution of the constrained martingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, B, \overline{\Xi})$ starting at $x = 0$. Fix $0 < \delta < 1$, and let

$$T^1 := \inf\{t \geq 0 :\ \Psi(X(t)) \geq 1\},$$
$$X^1(t) := X(t\wedge T^1).$$

Define

$$\vartheta^1_0 := \inf\{t \geq 0 :\ X^1(t) = 0\},$$
$$\theta^1_n := \inf\{t \geq \vartheta^1_{n-1} :\ \Psi(X^1(t)) = \delta\}, \quad n \geq 1,$$
$$\vartheta^1_n := \inf\{t \geq \theta^1_n :\ X^1(t) = 0\}, \quad n \geq 1,$$

with the usual convention that the infimum of the empty set is ∞. By the continuity of X, $\theta^1_n \uparrow \infty$ and $\vartheta^1_n \uparrow \infty$ as $n \to \infty$. We have

$$\begin{aligned}
& \Psi^\epsilon(X^1(t)) \\
= \quad & \sum_{n=0}^{\infty} \mathbf{1}_{\{\vartheta^1_n \le t\}} [\Psi^\epsilon(X^1(t \wedge \theta^1_{n+1})) - \Psi^\epsilon(X^1(\vartheta^1_n))] \\
& \quad + \sum_{n=1}^{\infty} \mathbf{1}_{\{\theta^1_n \le t\}} [\Psi^\epsilon(X^1(t \wedge \vartheta^1_n)) - \Psi^\epsilon(X^1(\theta^1_n))].
\end{aligned} \tag{A1.20}$$

As far as the first summand is concerned, we have, by $(A1.19)$, on the set $\{\vartheta^1_n \le t\}$,

$$|\Psi^\epsilon(X^1(t \wedge \theta^1_{n+1}) - \Psi^\epsilon(X^1(\vartheta^1_n))| = \Psi^\epsilon(X^1(t \wedge \theta^1_{n+1}) \le c_2 \delta^{\alpha^{\epsilon *}/\alpha^*}.$$

In addition, it can be easily checked that the argument used to prove (52) in Williams (1985), combined with Lemma 2.8 of Kwon and Williams (1991), still works, that is

$$\mathbb{E}\big[\sum_{n=0}^{\infty} \mathbf{1}_{\{\vartheta^1_n \le t\}}\big] \le \mathbb{E}\big[\sum_{n=0}^{\infty} \mathbf{1}_{\{\theta^1_n \le t\}}\big] \le c\,\delta^{-1}\,\frac{t+1}{1-\delta^{2/\alpha^*}}, \tag{A1.21}$$

where c depends only on α^* and ψ_{α^*}. Thus, for each ϵ, by (**??**), the expectation of the first summand vanishes as $\delta \to 0$. As for the second summand in $(A1.20)$, by $(A1.21)$ and the definition of X^1, it is bounded above by an integrable random variable. Moreover, taking into account that $\theta^1_n \le t$ implies $\theta^1_n \le T^1$, hence $\{\theta^1_n \le t\} \in \mathcal{F}_{\theta^1_n \wedge T^1}$, we have

$$\begin{aligned}
& \mathbb{E}\big[\sum_{n=1}^{\infty} \mathbf{1}_{\{\theta^1_n \le t\}} [\Psi^\epsilon(X^1(t \wedge \vartheta^1_n)) - \Psi^\epsilon(X^1(\theta^1_n))]\big] \\
= \quad & \mathbb{E}\big[\sum_{n=1}^{\infty} \mathbf{1}_{\{\theta^1_n \le t\}} [\Psi^\epsilon(X(t \wedge \vartheta^1_n \wedge T^1)) - \Psi^\epsilon(X(\theta^1_n \wedge T^1))]\big] \\
= \quad & \mathbb{E}\Big[\sum_{n=1}^{\infty} \mathbf{1}_{\{\theta^1_n \le t\}} \mathbb{E}\big[\Psi^\epsilon(X(t \wedge \vartheta^1_n \wedge T^1)) - \Psi^\epsilon(X(\theta^1_n \wedge T^1))\big|\mathcal{F}_{\theta^1_n \wedge T^1}\big]\Big].
\end{aligned}$$

Let us show that

$$\mathbb{E}\big[\Psi^\epsilon(X(t\wedge\vartheta^1_n\wedge T^1)) - \Psi^\epsilon(X(\tau^1_n\wedge T^1))\big|\mathcal{F}_{\theta^1_n\wedge T^1}\big]\Big] = \mathbb{E}\int_{\left[\theta^1_n\wedge T^1, t\wedge\vartheta^1_n\wedge T^1\right]\times\Xi} B\Psi^\epsilon(x,u)\,\Lambda(ds\times dx\times du)\big|\mathcal{F}_{\theta^1_n\wedge T^1}\Big].$$

In fact, setting, for $0 < \eta < \delta$, $\vartheta^1_{\eta,n} := \inf\{t \ge \theta^1_n :\ \Psi(X(t)) = \eta\}$, by $(A1.19)$, we have

$$\mathbb{E}\big[\Psi^\epsilon(X(t\wedge\vartheta^1_{\eta,n}\wedge T^1)) - \Psi^\epsilon(X(\theta^1_n\wedge T^1))\big|\mathcal{F}_{\theta^1_n\wedge T^1}\big]\Big] = \mathbb{E}\int_{\left[\theta^1_n\wedge T^1, t\wedge\vartheta^1_{\eta,n}\wedge T^1\right]\times\Xi} B\Psi^\epsilon(x,u)\,\Lambda(ds\times dx\times du)\big|\mathcal{F}_{\theta^1_n\wedge T^1}\Big].$$

Sending η to zero, by the continuity of X, we obtain the desired result. Then we can continue the above chain of equalities with

$$
\begin{aligned}
&= \mathbb{E}\Big[\sum_{n=1}^{\infty} \mathbf{1}_{\{\theta_n^1 \le t\}} \int_{\big[\theta_n^1 \wedge T^1, t\wedge \vartheta_n^1 \wedge T^1\big]\times \bar{\Xi}} B\Psi^\epsilon(x,u)\,\Lambda(ds\times dx\times du)\Big] \\
&\le \mathbb{E}\Big[\int_{\big[0,t\wedge T^1\big]\times \bar{\Xi}} B\Psi^\epsilon(x,u)\,\Lambda(ds\times dx\times du)\Big].
\end{aligned}
$$

Summing up, we have proved that, for each ϵ,

$$
\mathbb{E}\big[\Psi^\epsilon(X(t\wedge T^1))\big] \le \mathbb{E}\Big[\int_{\big[0,t\wedge T^1\big]\times \bar{\Xi}} B\Psi^\epsilon(x,u)\,\Lambda(ds\times dx\times du)\Big].
$$

By $(A1.19)$, for $\Psi(x) \le 1$, $\Psi^\epsilon(x) \ge c_1\Psi(x)^2$, and $\nabla\Psi^\epsilon \cdot \bar{g}$ converges to zero uniformly on $\partial\mathcal{K} \cap \{x : \Psi(x) \le 1\}$. In addition, by Theorem A1.9, $\mathbb{E}[\Lambda([0,t\wedge T^1]\times \bar{\Xi})] < \infty$. Therefore, in the limit as $\epsilon \to 0$, we find

$$
\mathbb{E}\big[\Psi(X(t\wedge T^1))^2\big] = 0, \quad \forall t > 0,
$$

which contradicts the fact that X spends zero time at the origin. □

A2 Auxiliary functions

Lemma A2.1 *There exists $\delta^* > 0$ and a function $V \in \mathcal{C}^2(\overline{D} - \{0\})$ such that*

$$
V(x) > 0, \quad \text{for } x \in \big(\overline{D} - \{0\}\big), \quad \lim_{x\in(\overline{D}-\{0\}),\, x\to 0} V(x) = 0 \tag{A2.1}
$$

$$
\nabla V(x)\cdot g(x) \le 0, \quad \text{for } x \in \big(\partial D - \{0\}\big) \cap \overline{B_{\delta^*}(0)} \tag{A2.2}
$$

Proof. The proof is similar to that of Lemma 2.2 of Kwon and Williams (1991). Let α^*, ψ_{α^*} and ψ^0 be as in Theorem A1.2, for $d = 2$, and in Theorem th:funcd3, for $d \ge 3$, and let Ψ be given by $(A1.5)$. Since $\partial\mathcal{S}$ is smooth, by Condition 3.1 (ii), we can extend ψ_{α^*} to a $\mathcal{C}^2$ function on some open neighborhood $\mathcal{S}^*$ of $\overline{\mathcal{S}}$ such that

$$
\inf_{z\in\mathcal{S}^*} \psi_{\alpha^*}(z) > 0.
$$

Analogously we can extend ψ^0 to a $\mathcal{C}^2$ function on some open neighborhood $\mathcal{S}^*$ of $\overline{\mathcal{S}}$. Let $\mathcal{K}^* := \{x : x = rz,\, z \in \mathcal{S}^*,\, r > 0\}$.

Let Φ be the function defined in $(A1.9)$. We have

$$
\Phi(x) > 0 \text{ in } K^*, \quad \lim_{x\in\mathcal{K}^*,\, x\to 0} \Phi(x) = 0, \tag{A2.3}
$$

and, if $\alpha^* = 0$,

$$
0 < c'_\Phi \le |\nabla\Phi(x)| \le c_\Phi, \quad |D^2\Phi(x)| \le \frac{c_\Phi}{|x|}, \quad x \in \mathcal{K}^*, \tag{A2.4}
$$

if $\alpha^* \neq 0$,

$$\frac{c'_\Phi \Phi(x)}{|x|} \leq |\nabla\Phi(x)| \leq \frac{c_\Phi \Phi(x)}{|x|}, \quad |D^2\Phi(x)| \leq \frac{c_\Phi}{|x|^2}, \quad x \in \mathcal{K}^*. \tag{A2.5}$$

We will look for a function V of the form

$$V(x) := f(\Phi(x)) - c_V e \cdot x,$$

for some $f \in \mathcal{C}^2((0,\infty))$, so that

$$\nabla V(x) \cdot \bar{g}(x) \leq -c_V c_e, \quad \text{for } x \in \partial\mathcal{K} - \{0\}.$$

By Condition 3.1 (i), there is δ^*, $0 < \delta^* \leq r_D$, such that $\big(\overline{D} - \{0\}\big) \cap \overline{B_{\delta^*}(0)} \subset \mathcal{K}^* \cap \overline{B_{\delta^*}(0)}$. Then, for $x \in \big(\overline{D} - \{0\}\big) \cap \overline{B_{\delta^*}(0)}$, letting $\bar{z}$ be the closest point on $\partial\mathcal{S}$ to $\frac{x}{|x|}$, by Condition 3.1 (i) and Condition 3.3 (i), we have

$$\begin{aligned}
&\nabla V(x) \cdot g(x) \\
&\leq \nabla V(|x|\bar{z}) \cdot \bar{g}(|x|\bar{z}) \,+\, |\nabla V(x) - \nabla V(|x|\bar{z})| \,+\, |\nabla V(|x|\bar{z})|\,|g(x) - \bar{g}(|x|\bar{z})| \\
&\leq -c_V c_e \,+\, d \sup_{0<t<1} \big|D^2 V(tx + (1-t)|x|\bar{z})\big| c_D\,|x|^2 \,+\, |\nabla V(|x|\bar{z})| c_g\,|x|,
\end{aligned}$$

Therefore, in order to ensure $(A2.2)$ for some δ^*, it is enough to choose f so that

$$\lim_{x\in\mathcal{K}^*,\, x\to 0} |\nabla V(x)|\,|x| = 0, \quad \lim_{x\in\mathcal{K}^*,\, x\to 0} |D^2 V(x)|\,|x|^2 = 0$$

(note that, for $|x| \leq \delta^*$, $\delta^* \leq \sqrt{3}/c_D$, it holds $\inf_{0<t<1} \big|tx + (1-t)|x|\bar{z}\big| \geq \frac{1}{2}|x|$). In view of $(A2.4)$ and $(A2.5)$, this is implied by

$$\lim_{x\in\mathcal{K}^*,\, x\to 0} |f'(\Phi(x))|\,|\nabla\Phi(x)|\,|x| = 0,$$

$$\lim_{x\in\mathcal{K}^*,\, x\to 0} |f''(\Phi(x))|\,|\nabla\Phi(x)|^2\,|x|^2 = 0, \quad \lim_{x\in\mathcal{K}^*,\, x\to 0} |f'(\Phi(x))|\,|D^2\Phi(x)|\,|x|^2 = 0. \tag{A2.6}$$

If, in addition,

$$\inf_{x\in\mathcal{K}^*\cap\overline{B_{\delta^*}(0)}} \frac{f(\Phi(x))}{|x|} > 0, \tag{A2.7}$$

then, by choosing $c_V = \frac{1}{2}\inf_{x\in\mathcal{K}^*} \frac{f(\Phi(x))}{|x|}$, we will obtain $V(x) > 0$ for $x \in \big(\overline{D} - \{0\}\big) \cap \overline{B_{\delta^*}(0)}$. Since we can always extend V to a strictly positive function in $\mathcal{C}^2\big(\big(\overline{D} - \{0\}\big)\big)$, $(A2.1)$ will be be satisfied.

Therefore we can take

$$f(u) := u^{1/|\alpha^*|}, \text{ for } \alpha^* \neq 0, \qquad f(u) := u, \text{ for } \alpha^* = 0.$$

□

Proof of Lemma 3.14.

Let δ^* and V be as in Lemma A2.1.

By Condition 3.3 (i) and (iv), possibly by taking a smaller δ^*, we can always suppose that

$$\inf_{g\in G(x),|g|=1,\, x\in\overline{D},\, |x|\leq\delta^*} e\cdot g > 0.$$

Let $0 < p^* < 1$ be such that

$$\sup_{x\in\overline{D},\, |x|\leq p^*\delta^*} V(x) < \inf_{x\in\overline{D},\, |x|\geq\delta^*} V(x).$$

Let $\widetilde{D}$ be a bounded domain with $\mathcal{C}^1$ boundary such that $\widetilde{D}\subset D$ and $\overline{\widetilde{D}}\cap\Big(B_{p^*\delta^*}(0)\Big)^c = \overline{D}\cap\Big(B_{p^*\delta^*}(0)\Big)^c$ and let $\widetilde{g}:\mathbb{R}^d\to\mathbb{R}^d$ be a locally Lipschitz vector field, of unit length on $\partial\widetilde{D}$, such that $\widetilde{g}(x)=g(x)$ for $x\in\partial\widetilde{D}\cap\Big(B_{p^*\delta^*}(0)\Big)^c$ and, denoting by $\widetilde{n}(x)$ the unit, inward normal at $x\in\partial\widetilde{D}$, it holds $\inf_{x\in\partial\widetilde{D}}\widetilde{g}(x)\cdot\widetilde{n}(x)>0$. There exists a function $\widetilde{F}\in\mathcal{C}^2(\overline{\widetilde{D}})$ such that

$$\inf_{x\in\partial\widetilde{D}}\nabla\widetilde{F}(x)\cdot\widetilde{g}(x)>0,$$

(see, e.g. Crandall et al. (1992), Lemma 7.6). Of course we can always assume that

$$\sup_{x\in\overline{D},\ p^*\delta^*\leq|x|\leq\delta^*}\widetilde{F}(x)\leq-\delta^*.$$

Now let $\chi:\mathbb{R}\to[0,1]$ be a nonincreasing, $\mathcal{C}^\infty$ function such that $\chi(u)=1$ for $u\leq\sup_{x\in\overline{D},\,|x|\leq p^*\delta^*}V(x)$ and $\chi(u)=0$ for $u\geq\inf_{x\in\overline{D},\,|x|\geq\delta^*}V(x)$. Defining

$$F(x):=\chi(V(x))\,e\cdot x+\big(1-\chi(V(x))\big)\widetilde{F}(x),$$

we have

$$\nabla F(x)=\big[\chi(V(x))\,e+\big(1-\chi(V(x))\big)\nabla F^*(x)\big]+\big(e\cdot x-\widetilde{F}(x)\big)\chi'(V(x))\nabla V(x)$$

so that, for all $x\in\partial D-\{0\}$,

$$\nabla F(x)\cdot g(x)\geq\inf_{g\in G(x),|g|=1,\, x\in\overline{D},\, |x|\leq\delta^*} e\cdot g\wedge\inf_{x\in\partial\widetilde{D}}\nabla\widetilde{F}(x)\cdot\widetilde{g}(x).$$

$\square$

Proof of Lemma A1.8. Consider the function Φ defined in $(A1.9)$ and let $\delta>1$ be such that $\sup_{x\in\overline{\mathcal{K}},\,|x|\leq1}\Phi(x)<\inf_{x\in\overline{\mathcal{K}},\,|x|\geq\delta}\Phi(x)$. Let D be a bounded domain such that $D\subset\mathcal{K}\cap B_{\delta+1}(0)$, $\overline{D}\cap\overline{B_\delta(0)}=\overline{\mathcal{K}}\cap\overline{B_\delta(0)}$ and $\partial D-\{0\}$ is of class $\mathcal{C}^1$. Let $g:\mathbb{R}^d\to\mathbb{R}^d$ be a locally Lipschitz vector field, of unit length on ∂D, such that $g(x)=\bar{g}(x)$ for $x\in\partial D\cap\overline{B_\delta(0)}$ and, denoting by $n(x)$ the unit, inward normal at $x\in\partial D$, it holds $\inf_{x\in\partial D-\{0\}}g(x)\cdot n(x)>$

0. Then Lemma 3.14 ensures the existence of a function $F^D \in \mathcal{C}^2(\overline{\mathcal{K}} \cap \overline{B_\delta(0)})$ such that $\nabla F^D(x) \cdot \bar{g}(x) \geq c_{F^D} > 0$ for every $x \in \big(\partial\mathcal{K} - \{0\}\big) \cap \overline{B_\delta(0)}$.

On the other hand, there exists a function $F^{\mathcal{S}} \in \mathcal{C}^2(\overline{\mathcal{S}})$ such that $\nabla_{S^{d-1}} F^{\mathcal{S}}(z) \cdot \bar{g}(z) \geq c_{F^{\mathcal{S}}} > 0$ for every $z \in \partial\mathcal{S}$. Define $F^{\mathcal{S}}(x)$ for $x \in \overline{\mathcal{K}} - \{0\}$ as $F^{\mathcal{S}}(x) := F^{\mathcal{S}}(\frac{x}{|x|})$.

Let $\chi : \mathbb{R} \to [0,1]$ be a nonincreasing, $\mathcal{C}^\infty$ function such that $\chi(u) = 1$ for $u \leq \sup_{x\in\overline{\mathcal{K}},\,|x|\leq 1} \Phi(x)$ and $\chi(u) = 0$ for $u \geq \inf_{x\in\overline{\mathcal{K}},\,|x|\geq\delta} \Phi(x)$. Then the function

$$F(x) := \chi(\Phi(x))\, F^D(x) + \big(1 - \chi(\Phi(x))\big) F^{\mathcal{S}}(x)$$

has the desired properties. □

Proof of Lemma 3.25.

Let α^*, ψ_{α^*} and ψ^0 be as in Theorem A1.2, for $d = 2$, and in Theorem th:funcd3, for $d \geq 3$, and let Ψ be given by $(A1.5)$. By Theorem A1.10, we can fix $\beta > 0$ such that

$$\alpha^* < \beta < 1. \tag{A2.8}$$

Then, by Remark A1.7, we can extend ψ_{α^*} to a $\mathcal{C}^{2+\beta}$ function on some open neighborhood $\mathcal{S}^*$ of $\overline{\mathcal{S}}$ such that

$$\inf_{z\in\mathcal{S}^*} \psi_{\alpha^*}(z) > 0.$$

Analogously we can extend ψ^0 to a $\mathcal{C}^{2+\beta}$ function on some open neighborhood $\mathcal{S}^*$ of $\overline{\mathcal{S}}$. Let $\mathcal{K}^* := \{x :\, x = rz,\, z \in \mathcal{S}^*,\, r > 0\}$. We will choose $\mathcal{S}^*$ such that

$$e \cdot x \geq -\widetilde{c}_e|x|, \quad 0 < \widetilde{c}_e < 1, \qquad x \in \mathcal{K}^*. \tag{A2.9}$$

Note that the derivatives of Ψ satisfy the same bounds as Φ ($(A2.4)$ and $(A2.5)$). Moreover, $(A2.8)$, combined with $(A1.6)$ and Condition 3.1 (i), implies that, for $\delta^* \leq r_D$,

$$|\Delta\Psi(x)| \leq \frac{c_\Psi}{|x|^{2-\beta}}, \quad x \in \overline{D} \cap \overline{B_{\delta^*}(0)}, \tag{A2.10}$$

if $\alpha^* = 0$,

$$|\Delta\Psi(x)| \leq c_\Psi \frac{\Psi(x)}{|x|^{2-\beta}}, \quad x \in \overline{D} \cap \overline{B_{\delta^*}(0)}, \tag{A2.11}$$

if $\alpha^* \neq 0$, Consider first the case $\alpha^* \leq 0$. We look for V of the form

$$V(x) := f(\Psi(x)) - e \cdot x.$$

By the same computations as in the proof of Lemma A2.1, we see that (3.21) is verified as soon as $(A2.6)$ holds (with ϕ replaced by Ψ, of course). As far as (3.22) is concerned, we have, by Condition 3.6,

$$\begin{aligned} &AV(x) \\ &\leq \frac{1}{2}\Delta V(x) \,+\, |b(x)|\,|\nabla V(x)| \,+\, \frac{1}{2}|(\sigma\sigma^T)(x) - I|\,|D^2V(x)| \end{aligned}$$

and, supposing $(A2.6)$ holds (with ϕ replaced by Ψ),

$$\begin{aligned}
&\leq \frac{1}{2}\Delta V(x) + |x|^{-1}o(1) \\
&= f''(\Psi(x))\,|\nabla\Psi(x)|^2 + f'(\Psi(x))\,\Delta\Psi(x) + |x|^{-1}o(1) \\
&\leq |x|^{\beta-2}\big(f''(\Psi(x))\,|\nabla\Psi(x)|^2|x|^{2-\beta} + f'(\Psi(x))\,\Delta\Psi(x)|x|^{2-\beta} + |x|^{1-\beta}o(1)\big).
\end{aligned}$$

Hence, taking into account $(A2.4)$ and $(A2.5)$ (with Φ replaced by Ψ) and $(A2.11)$, (3.22) holds if

$$\sup_{x\in\overline{D}\cap\overline{B_{\delta^*}(0)}} f''(\Psi(x)) < 0,$$

and

$$\lim_{x\in\overline{D},\,x\to 0} f'(\Psi(x)) = 0, \text{ if } \alpha^* = 0, \qquad \lim_{x\in\overline{D},\,x\to 0} f'(\Psi(x))\,\Psi(x) = 0, \text{ if } \alpha^* < 0.$$

Therefore we can take

$$f(u) := \ln(u), \text{ for } \alpha^* = 0, \qquad f(u) := \ln(\ln(u)), \text{ for } \alpha^* < 0.$$

With these choices, also (3.20) is verified.

In the case $0 < \alpha^* < 1$, one can check, by computations analogous to those above, that we can take

$$V_1(x) = \exp(\Psi(x)) - 1 + e\cdot x, \quad V_2(x) = \ln(\Psi(x)+1) - e\cdot x.$$

□

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