Math 281C Homework 2 Solutions

Throughout the solutions, uppercase letters denote random variables, and lowercase letters denote realized values.

1. Suppose X is one observation from a population with beta$(\theta, 1)$ pdf—$Cx^{\theta-1}$ for $0 < x < 1$.

   (a) For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the size and sketch the power function of the test that rejects H_0 if $X > 1/2$.

   Solution: It follows from $\int_0^1 Cx^{\theta-1}\mathrm{d}x = 1$ that $C = \theta$. For any $\theta > 0$,

   $$\mathbb{P}(X > 1/2|\theta) = 1 - (1/2)^\theta,$$

   so the size is $\sup_{\theta\leq 1} \mathbb{P}(X > 1/2|\theta) = 1/2$. The power curve can be accordingly drawn.

   (b) Find the most powerful level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

   Solution: By the N-P Lemma, we reject H_0 if

   $$\lambda = \frac{f(x|\theta = 2)}{f(x|\theta = 1)} = 2x > c.$$

   The value c satisfies $\alpha = \mathbb{P}(2X > c|\theta = 1)$, which yields $c = 2(1 - \alpha)$.

   (c) Is there a UMP test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$? If so, find it; if not, prove so.

   Solution: Yes. It can be shown that the distribution has MLR in x, and the existence of UMP is guaranteed by Theorem 3.2.1. The test rejects H_0 if $x \geq 1 - \alpha$.

2. Let X be one observation from a Cauchy scale distribution with density

   $$f_\theta(x) = \frac{\theta}{\pi}\frac{1}{\theta^2 + x^2}, \quad -\infty < x < \infty, \theta > 0.$$

   (a) Show that this family does not have an MLR in x.

   Solution: For any $\theta_2 > \theta_1$,

   $$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2}{\theta_1}\frac{\theta_1^2 + x^2}{\theta_2^2 + x^2},$$

   and its derivative depends on the sign of x, so the above function is not monotonic in x.

   (b) Show that the distribution of $|X|$ does have an MLR.

   Solution: By symmetry, the density of $|X|$ is

   $$f_\theta(x) = \frac{2\theta}{\pi}\frac{1}{\theta^2 + x^2}, \quad 0 \leq x < \infty, \theta > 0.$$

   The likelihood ratio can be similarly calculated, and is monotonic in x.

3. Let X be one observation from a Cauchy distribution

   $$f_\theta(x) = \frac{C}{1 + (x - \theta)^2}, \quad x \in \mathbb{R}.$$

   (a) Show that this family does not have an MLR in x.

   Solution: For any $\theta_2 > \theta_1$,

   $$\lambda(x|\theta_2, \theta_1) := \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{1 + (x - \theta_1)^2}{1 + (x - \theta_2)^2}.$$

   We have $\lim_{x\to-\infty} \lambda(x|\theta_2, \theta_1) = \lim_{x\to\infty} \lambda(x|\theta_2, \theta_1) = 1$, and $\lambda(0) \neq 1$, so $\lambda(x|\theta_2, \theta_1)$ is not monotonic in x. Alternatively, we can calculate its derivative to reach the same conclusion.

(b) Show that the test

$$\phi(x) = \mathbb{1}(1 < x < 3)$$

is UMP of its size for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Calculate the Type I and type II error probabilities.

Solution: By the N-P Lemma, the UMP test rejects H_0 if

$$\lambda(x) = \frac{1 + x^2}{1 + (x-1)^2} > c,$$

and such test is unique. It remains to check that $\lambda(x) > c$ is equivalent to $1 < x < 3$, which can be done by checking the derivative of $\lambda(x)$ and $\lambda(1) = \lambda(3)$.
Finally, the Type I error probability is $\mathbb{P}(1 < X < 3|\theta = 0) = 0.1476$, and the Type II error probability is $\mathbb{P}(X \le 1 \text{ or } X \ge 3|\theta = 1) = 0.6476$.

4. Let $X_1, \ldots, X_n$ be i.i.d. from the exponential family

$$f_\theta(\mathbf{x}) = \exp\{\eta(\theta)T(\mathbf{x}) - \xi(\theta)\}h(\mathbf{x}),$$

where $\eta(\theta)$ is strictly monotone in θ. Show that UMP tests do not exist for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \ne \theta_0$. Hint: Examine the example in Section 4.2.

Solution: WLOG, we assume that $\eta(\theta)$ is strictly increasing in θ. For any size α and $\theta_1 < \theta_0$, the test $\phi_1(\boldsymbol{x}) = \mathbf{1}\{\sum_{i=1}^n T(x_i) < c_1\}$ has the highest power at $\theta = \theta_1$, where c_1 satisfies $\alpha = \mathbb{P}(\sum_{i=1}^n T(X_i) < c_1|\theta = \theta_0)$, and if a UMP test exists for all $\theta_1 \ne \theta_0$, then it must be ϕ_1. However, for any $\theta_1 > \theta_0$, consider another test $\phi_2(\boldsymbol{x}) = \mathbf{1}\{\sum_{i=1}^n T(x_i) > c_2\}$ where c_2 satisfies $\alpha = \mathbb{P}(\sum_{i=1}^n T(X_i) > c_2|\theta = \theta_0)$, and it follows from the N-P Lemma that ϕ_2 is more powerful than ϕ_1. This is a contradiction.

5. (optional) Let $X_1, \ldots, X_n$ be a random sample from Unif$(\theta, \theta+1)$ distribution. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, use the test

$$\text{reject } H_0 \quad \text{if } Y_n \ge 1 \text{ or } Y_1 \ge k,$$

where k is a constant, $Y_1 = \min_{1\le i\le n} X_i$ and $Y_n = \max_{1\le i\le n} X_i$. Hint: find the marginal distribution of Y_1, and the joint distribution of (Y_1, Y_n).

(a) Determine k so that the test has size α.

Solution: It follows from

$$\alpha = \mathbb{P}(Y_1 \ge k \text{ or } Y_n \ge 1|\theta = 0) = (1-k)^n$$

that $k = 1 - \alpha^{1/n}$.

(b) Find an expression for the power function of the test in part (a).

Solution: The PDF of Y_1, Y_n are

$$f_{Y_1}(y) = n(\theta + 1 - y)^{n-1}, \quad f_{Y_n}(y) = n(y-\theta)^{n-1}, \qquad \theta \le y \le \theta + 1,$$

and the joint PDF of (Y_1, Y_n) is

$$f_{Y_1,Y_n}(y_1, y_n) = n(n-1)(y_n - y_1)^{n-2}, \qquad \theta \le y_1 \le y_n \le \theta + 1.$$

If $0 < \theta \le k$,

$$\begin{aligned}
\mathbb{P}(Y_1 \ge k \text{ or } Y_n \ge 1|\theta) &= \mathbb{P}(Y_1 \ge k|\theta) + \mathbb{P}(Y_n \ge 1|\theta) - \mathbb{P}(Y_1 \ge k, Y_n \ge 1|\theta) \\
&= \mathbb{P}(Y_1 \ge k|\theta) + \mathbb{P}(Y_1 < k, Y_n \ge 1|\theta) \\
&= 1 + \alpha - (1-\theta)^n.
\end{aligned}$$

If $\theta \ge k$,

$$\mathbb{P}(Y_1 \ge k \text{ or } Y_n \ge 1|\theta) = 1.$$