Math 281C Homework 6 Solutions

1. Find the LRT for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ based on a single observation from the density function
$$f(x) = 2\frac{\theta - x}{\theta^2}\mathbb{1}(0 < x < \theta).$$

   Solution: Viewing the density as a function of θ and setting $\partial f_x(\theta)/\partial\theta = 0$ gives us $\widehat{\theta} = 2x$. The LRT statistic is then
$$\lambda(x) = \frac{\ell(\theta_0)}{\ell(\widehat{\theta})} = 4\frac{(\theta_0 - x)x}{\theta_0^2}\mathbb{1}(0 < x < \theta_0).$$
   The LRT rejects H_0 if $\lambda(x) < C$, which is equivalent to $x < (\theta_0 - \theta_0\sqrt{1-C})/2$ or $x > (\theta_0 + \theta_0\sqrt{1-C})/2$. If the test size is α, then the constant C can be determined by $\mathbb{P}\{x < (\theta_0 - \theta_0\sqrt{1-C})/2\} + \mathbb{P}\{x > (\theta_0 - \theta_0\sqrt{1-C})/2\} = \alpha$.

2. Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be two independent samples with the probability density functions
$$f_1(x) = \frac{1}{\lambda_1}e^{-x/\lambda_1}\mathbb{1}(x > 0) \quad \text{and} \quad f_2(y) = \frac{1}{\lambda_2}e^{-y/\lambda_2}\mathbb{1}(y > 0),$$
   respectively. We wish to test $H_0 : \lambda_1 = \lambda_2$ versus $H_1 : \lambda_1 \neq \lambda_2$.

   (i). Find a UMPU test of size α.

   Solution: The joint density is
$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{y}) &= \frac{1}{\lambda_1^m\lambda_2^n}\exp\left\{-\frac{\sum_{i=1}^m x_i}{\lambda_1} - \frac{\sum_{i=1}^n y_i}{\lambda_2}\right\} \cdot \mathbb{1}(X_{(1)} > 0, Y_{(1)} > 0) \\ &= \frac{1}{\lambda_1^m\lambda_2^n}\exp\left\{(1/\lambda_2 - 1/\lambda_1)\sum_{i=1}^m x_i - (1/\lambda_2)\Big(\sum_{i=1}^m x_i + \sum_{i=1}^n y_i\Big)\right\} \cdot \mathbb{1}(X_{(1)} > 0, Y_{(1)} > 0) \end{aligned}$$
   The following argument is similar to Question 2-(i) in homework 4. Denote $U := \sum_{i=1}^m X_i$, $T := \sum_{i=1}^m x_i + \sum_{i=1}^n y_i$, and define $W = h(U, T) := U/T$. When $\lambda_1 = \lambda_2 = \lambda$, T is sufficient and complete. We then show W is ancillary. Notice that $X_i \sim \text{Gamma}(\lambda, 1)$ and $Y_i \sim \text{Gamma}(\lambda, 1)$, so $U \sim \text{Gamma}(\lambda, m)$, $T \sim \text{Gamma}(\lambda, m+n)$, and $W \sim \text{Beta}(m, n)$. Hence, the distribution of W does not depend on λ. Applying Basu's Theorem gives us the independence between W and T.
   The UMPU test can be obtained by applying Theorem 6.2.1. Notice that $h(u, t)$ is linear in u for each t, so a UMPU test of size α takes the form
$$\phi(w) = \begin{cases} 1 & \text{when } w \leq c_1 \text{ or } w \geq c_2, \\ 0 & \text{when } c_1 < w < c_2, \end{cases}$$
   where c_1, c_2 satisfy $\mathbb{E}\phi(W) = \alpha$ and $\mathbb{E}[W\phi(W)] = \alpha\mathbb{E}W$, where $W \sim \text{Beta}(m, n)$.

   (ii). Find an LRT of size α.

   Solution: In the null space, $\widehat{\lambda}_1 = \widehat{\lambda}_2 = \overline{X + Y}$, and in the whole space, $\widehat{\lambda}_1 = \overline{X}, \widehat{\lambda}_2 = \overline{Y}$. The LRT statistic is
$$\lambda(\boldsymbol{x}, \boldsymbol{y}) = \left(\frac{\overline{X}}{\overline{X + Y}}\right)^m \left(\frac{\overline{Y}}{\overline{X + Y}}\right)^n \cdot \mathbb{1}(X_{(1)} > 0, Y_{(1)} > 0).$$
   The LRT rejects H_0 if $\lambda(\boldsymbol{x}, \boldsymbol{y}) < C$ for some constant C.

(iii). Are the two tests in (i) and (ii) the same?

Solution: Yes. First notice that

$$\lambda(\boldsymbol{x}, \boldsymbol{y}) = \frac{(m+n)^{m+n}}{m^m n^n} W^m (1-W)^n,$$

where $W = \sum_{i=1}^m X_i / (\sum_{i=1}^m x_i + \sum_{i=1}^n y_i)$ is defined in part (i). For any values of (m, n), $f(x) = x^m(1-x)^n$ is first increasing then decreasing over $x \in (0,1)$, so $\lambda(\boldsymbol{x}, \boldsymbol{y}) < C$ is equivalent to $W < c_3$ or $W > c_4$, and c_3, c_4 satisfy the condition

$$c_3^m (1-c_3)^n = c_4^m (1-c_4)^n. \tag{1}$$

In the following, we show (c_1, c_2) also satisfy the above condition, where c_1, c_2 are the constants in part (i).
The constants c_1, c_2 satisfy

$$\int_{c_1}^{c_2} \frac{x^{m-1}(1-x)^{n-1}}{B(m,n)} \mathrm{d}x = (1-\alpha) \quad \text{and} \quad \int_{c_1}^{c_2} \frac{x^{m}(1-x)^{n-1}}{B(m,n)} \mathrm{d}x = (1-\alpha)\frac{m}{m+n}.$$

Applying integration by parts to the left-hand side, we have

$$\int_{c_1}^{c_2} \frac{x^{m}(1-x)^{n-1}}{B(m,n)} \mathrm{d}x = \underbrace{-\frac{1}{n}\frac{x^m(1-x)^n}{B(m,n)}\Bigg|_{c_1}^{c_2}}_{\text{I}} + \underbrace{\frac{m}{n}\int_{c_1}^{c_2} \frac{x^{m-1}(1-x)^{n}}{B(m,n)} \mathrm{d}x}_{\text{II}}.$$

In addition,

$$\begin{aligned}
\text{II} &= \frac{m}{n}\int_{c_1}^{c_2} (1-x)\frac{x^{m-1}(1-x)^{n-1}}{B(m,n)} \mathrm{d}x \\
&= \frac{m}{n}\underbrace{\int_{c_1}^{c_2} \frac{x^{m-1}(1-x)^{n-1}}{B(m,n)} \mathrm{d}x}_{1-\alpha} - \frac{m}{n}\underbrace{\int_{c_1}^{c_2} \frac{x^{m}(1-x)^{n-1}}{B(m,n)} \mathrm{d}x}_{(1-\alpha)\frac{m}{m+n}} \\
&= (1-\alpha)\frac{m}{m+n},
\end{aligned}$$

which implies $\text{I} = 0$. This shows (c_1, c_2) satisfy equation (1), and hence proves the equivalence.