Math 281C Homework 7

Due: 5:00pm, May 20th

1. We have already seen the usefulness of the LRT in dealing with problems with nuisance parameters. We now look at another nuisance parameter problem. Find the LRT of size α for testing

$$H_0 : \gamma = 1 \quad \text{versus} \quad H_1 : \gamma \neq 1$$

based on a sample $X_1, \dots, X_n$ from the Weibull(γ, β) with pdf

$$f(x) = \frac{\gamma}{\beta} x^{\gamma - 1} e^{-x^\gamma / \beta}, \quad x > 0, \quad \beta > 0.$$

2. Consider a linear regression model $Y = \beta_0 + \beta_1 X + \sigma\varepsilon$, where β_0 is the intercept, β_1 is the slope coefficient, $\sigma > 0$ is the residual standard deviation, and ε is the (unobservable) random error satisfying $\varepsilon | X \sim N(0, 1)$. Assume $\beta_0, \beta_1, \sigma^2$ are all unknown. Find the LRT of size α for testing

$$H_0 : \beta_1 = 0 \quad \text{versus} \quad H_1 : \beta_1 \neq 0$$

based on independent observations $(X_1, Y_1), \dots, (X_n, Y_n)$ from (X, Y).

3. A random sample $X_1, \dots, X_n$ is drawn from a Pareto population with pdf

$$f(x) = \frac{\theta \nu^\theta}{x^{\theta + 1}} \mathbb{1}(x \geq \nu),$$

where $\theta, \nu > 0$.

(a) Find the MLEs of θ and ν.

(b) Show that the LRT of

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta \neq 1$$

has critical region of the form $\{\mathbf{x} : T(\mathbf{x}) < C_1 \text{ or } T(\mathbf{x}) > C_2\}$, where $0 < C_1 < C_2$ and

$$T = \log \left\{ \prod_{i=1}^{n} X_i / X_{(1)} \right\}.$$

(c) Show that, under H_0, $2T$ has a chi-squared distribution, and find the number of degrees of freedom. (Hint: Obtain the joint distribution of the $n - 1$ nontrivial terms $X_i / X_{(1)}$ conditional on $X_{(1)}$. Put these $n - 1$ terms together, and notice that the distribution of T given $X_{(1)}$ does not depend on $X_{(1)}$, and hence is also the unconditional distribution of T.)