Towards a Hodge-Iwasawa Theory

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Motivation I: Dememorization and Memorization

- Consider the cyclotomic tower \( \{ \mathbb{Q}_p(\zeta_{p^n}) \}_n \) of \( \mathbb{Q}_p \).
- The infinite level of this tower is kind of special after the corresponding completion.
- Over \( \mathbb{Q}_p \), we could consider \( \text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})_{\text{pro ét}} \) due to Scholze, although the infinite level of the towers above participates in this topology but the corresponding pro-étale site forgets the corresponding cyclotomic tower while it is defined by using pro-systems of étale morphisms.
- Work of Pottharst, Kedlaya-Pottharst-Xiao, Kedlaya-Pottharst implies one may see the corresponding cyclotomic tower back by considering the corresponding cyclotomic deformation as below.
- One has the so-called \( \psi \)-cohomology originally dated back to Fontaine attached to a \((\varphi, \Gamma)\)-module \( M \) (you could regarded this as a Galois representation):

\[
H_\psi(M)
\]  

(0.1)

by using the operator \( \psi \).

- And we have the corresponding \((\varphi, \Gamma)\)-module after Herr, but we consider the cyclotomic deformation as in Kedlaya-Pottharst-Xiao over the Robba ring \( \mathcal{R}^\infty_{\mathbb{Q}_p}(\Gamma) \):

\[
H_{\varphi, \Gamma}(\text{CycDef}(M)).
\]  

(0.2)
Motivation I: Dememorization and Memorization

- This is defined by taking the corresponding external tensor product of $M$ with the corresponding module coming from the quotient $\Gamma$. This dates back to Pottharst on his analytic Iwasawa cohomology.

- Work of Kedlaya-Pottharst observes that we can have the following sheaf version of the construction:

$$H_{\text{pro-\acute{e}tale}}(\text{Spa}(\mathbb{Q}_p, o_{\mathbb{Q}_p}), \text{CycDef}(\tilde{M})), \tag{0.3}$$

which is defined by taking the corresponding external product of Kedlaya-Liu’s sheaf $\tilde{M}$ with the one defined by using the quotient $\Gamma$.

- The point is that we have the following comparison:

$$H_{\psi}(M) \sim H_{\varphi, \Gamma}(\text{CycDef}(M)) \sim H_{\text{pro-\acute{e}tale}}(\text{Spa}(\mathbb{Q}_p, o_{\mathbb{Q}_p}), \text{CycDef}(\tilde{M})). \tag{0.4}$$

- Suppose $M(V)$ comes from a Galois representation $V$ of $G_{\mathbb{Q}_p}$ we even have the following comparison after Perrin-Riou:

$$\mathcal{O}_{\text{SpR}_{\mathbb{Q}_p}^\infty(\Gamma)} \hat{\otimes}_{\Lambda} H_W(G_{\mathbb{Q}_p}, V) \sim \tag{0.5}$$

$$\sim H_{\psi}((M(V))) \sim H_{\varphi, \Gamma}(\text{CycDef}((M(V))) \sim H_{\text{pro-\acute{e}tale}}(\text{Spa}(\mathbb{Q}_p, o_{\mathbb{Q}_p}), \text{CycDef}((\tilde{M}(V)))). \tag{0.6}$$

- Natural questions come:

I. How about the Lubin-Tate Iwasawa theory in Berger-Fourquaux-Schneider-Venjakob's work, observed by Kedlaya-Pottharst.

II. How about higher dimensional toric towers and more general towers of rigid analytic spaces for instance.
Motivation I: Dememorization and Memorization

- These need us to generalize the corresponding framework to higher dimensional situation and more general deformed version. The problem is challenging, since we have some rigidized objects combined together.

- Rational coefficients are very complicated comparing to algebraic geometry, since sometimes we do not have the integral lattices over the étale sites. This is already a problem in the context of Kedlaya-Liu.

- It is not surprising much for us to consider generalizing the framework of non-étale objects since even in the usual situations over a point work of Nakamura, Kedlaya-Potthast-Xiao and Kedlaya-Liu implies that all kinds of families of Galois representations will be more conveniently studied by using $B$-pairs and $(\varphi, \Gamma)$-modules.

- If we only have some abelian group $G$ the corresponding deformation happens along the algebra $\mathbb{Q}_p[G]$ which gives rise to Galois representation of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with coefficient in $\mathbb{Q}_p[G]$ along the quotient $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to G$. One can then regard this as a sheaf of module over some sheaf with deformed coefficient in $\mathbb{Q}_p[G]$.

- Note that we can also consider some deformation over an affinoid algebra in the rigid analytic geometry, which amounts to the $p$-adic families of special values. This is not available at once in archimedean functional analysis.
The corresponding equivariant consideration could be obviously generalized to the relative $p$-adic Hodge theory which is aimed at the study of the étale local systems over rigid analytic spaces.

This amounts to higher dimensional modeling of the generalized Weil conjecture after the work due to many people, to name a few Deligne ($\ell$-adic étale sheaves), Kedlaya ($p$-adic differential modules), Abe and Caro ($p$-adic arithmetic $D$-modules) and so on.

The invariance comes from the quotient of étale fundamental groups of rigid analytic spaces, or the corresponding profinite fundamental groups of rigid analytic spaces.

(Example) One can consider the corresponding Fréchet-Stein algebras associated to the group $\mathbb{Z}_p \times \mathbb{Z}_p^n$ which is Galois group (quotient of the corresponding profinite fundamental group) of a local chart of smooth proper rigid analytic spaces. Note that the top of this local chart in the smooth proper setting naturally participates in some nice topology.

(Example) One can consider the local systems in more general sense, for instance the locally constant sheaf $A$ attached to a topological ring, for instance an affinoid algebra in the rigid analytic geometry after Tate. This is somewhat special in the $p$-adic setting due to the fact the corresponding Hodge structures could achieve variation in $p$-adic rigid family.

We also want to mention the work of Deligne-Laumon, Abe-Marmora and Guignard.
Following some idea in the noncommutative Tamagawa Number conjecture after Fukaya-Kato and the noncommutative Iwasawa theory over a scheme over finite field after Witte we would like to consider the following picture after Witte.

Let $T$ be an adic ring in the sense of Fukaya-Kato, which is a compact ring with two sided ideal $I$ such that we have each $T/I^n$ is finite for $n \geq 0$ and taking the inverse limit we recover the ring $T$ itself. This ring could be noncommutative, for instance the Iwasawa algebra attached to some $p$-adic Lie group.

(Definition, after Witte) Consider a rigid analytic space or a scheme $X/\mathbb{Q}_p$ separated and of finite type we consider the category $\mathbb{D}_{\text{perf}}(X_\# , T)$ ($\# = \text{ét}, \text{pro-ét}$) which is the category of the inverse limit of perfect complexes of abelian sheaves of left modules over quotients of $T$ by open two-sided ideals of $T$ which are $DG$-flat, parametrized by open two-sided ideals of $T$.

(Theorem, Witte) Let $p$ be a unit in $T$. The category defined above could be endowed with the structure of Waldhausen category\(^1\) and the total direct image functor induces a well defined functor in the situation where $X$ is a scheme and $\# = \text{ét}:

\[
\mathbb{D}_{\text{perf}}(X_\#, T) \xrightarrow{R\Gamma(X_\#, \cdot)} \mathbb{D}_{\text{perf}}(T)
\]

which induces the corresponding map on the $K$-theory space:

\[
\mathbb{K}\mathbb{D}_{\text{perf}}(X_\#, T) \xrightarrow{\mathbb{K}R\Gamma(X_\#, \cdot)} \mathbb{K}\mathbb{D}_{\text{perf}}(T).
\]

Then this map is homotopic to zero in some canonical way.

\(^1\)Strictly Speaking, these are the complicial biWaldhausen ones.
Things discussed so far have motivated the corresponding equivariant relative $p$-adic Hodge Theory in the following sense. Witte considered general framework of Grothendieck abelian categories, for instance one can consider the following categories:

1. The ind-category of all the arithmetic $D$-modules over a realizable scheme of finite type over a perfect field $k$ as those considered by Berthelot, Caro, Abe and etc (this is not covered in our work).

2. The category of all the abelian sheaves over the étale or pro-étale sites of schemes of finite type over a field $k$ after Grothendieck, Scholze, Bhatt and etc;

3. The category of all the abelian sheaves over the étale or pro-étale sites of adic spaces of finite type over a field $k$ after Huber, Scholze, Kedlaya-Liu;

4. The ind-category of the abelian category of the pseudocoherent Frobenius $\varphi$-sheaves over a rigid analytic space over a complete discrete valued field with perfect residue field $k$ after Kedlaya-Liu.

One can naturally consider the corresponding $P$-objects throughout the categories listed above, where $P$ is noetherian for instance. For instance one can consider the third category and consider the corresponding local systems over $A$ where $A$ is an affinoid algebra in rigid analytic geometry after Tate, which are the $A$-objects in the corresponding category of all the abelian sheaves.
The corresponding $P$-objects are interesting, but in general are not that easy to study, especially we consider for instance those ring defined over $\mathbb{Q}_p$, let it alone if one would like to consider the categories of the complexes of such objects.

We choose to consider the corresponding embedding of such objects into the categories of Frobenius sheaves with coefficients in $P$ after Kedlaya-Liu. Again we expect everything will be more convenient to handle in the category of $(\varphi, \Gamma)$-modules.

Working over $R$ now a uniform Banach algebra with further structure of an adic ring over $\mathbb{F}_p$. And we assume that $R$ is perfect. Let $\tilde{\Pi}^I_R$ be the Robba sheaves defined by Kedlaya-Liu, with respect to some interval $I \subset (0, \infty)$, which are Fréchet completions of the ring of Witt vector of $R$ with respect to the Gauss norms induced from the norm on $R$.

Taking suitable interval one can define the corresponding Robba rings $\tilde{\Pi}^r_R$, $\tilde{\Pi}^\infty_R$ and the corresponding full Robba ring $\tilde{\Pi}_R$.

We work in the category of Banach and ind-Fréchet spaces, which are commutative. Our generalization comes from those Banach reduced affinoid algebras $A$. 
Equivariant relative $p$-adic Hodge Theory

- The $p$-adic functional analysis produces us some manageable structures within our study of relative $p$-adic Hodge theory, generalizing the original $p$-adic functional analytic framework of Kedlaya-Liu.
- Starting from Kedlaya-Liu’s period rings,

\[
\tilde{\Pi}_R^\infty, \tilde{\Pi}_R^I, \tilde{\Pi}_R^r, \tilde{\Pi}_R, \tilde{\Pi}_R^{\text{int}, r}, \tilde{\Pi}_R^{\text{int}, \text{bd}, r}, \tilde{\Pi}_R^{\text{bd}}
\]

(0.7)

we can form the corresponding $A$-relative of the period rings:

\[
\tilde{\Pi}_R^\infty, \tilde{\Pi}_R^I, \tilde{\Pi}_R^r, \tilde{\Pi}_R, \tilde{\Pi}_R^{\text{int}, r}, \tilde{\Pi}_R^{\text{int}, \text{bd}, r}, \tilde{\Pi}_R^{\text{bd}}
\]

(0.8)

- **(Remark)** There should be also many interesting contexts, for instance consider a finitely generated abelian group $G$, one can consider the group rings:

\[
\tilde{\Pi}_R^I[G].
\]

(0.9)

- And then consider the completion living inside the corresponding infinite direct sum Banach modules

\[
\bigoplus \tilde{\Pi}_R^I,
\]

(0.10)

over the corresponding period rings:

\[
\tilde{\Pi}_R^I[G].
\]

(0.11)

Then we take suitable intersection and union one can have possibly some interesting period rings $\tilde{\Pi}_R^I[G]$ and $\tilde{\Pi}_R[G]$. 
Equivariant relative $p$-adic Hodge Theory

- The equivariant period rings in the situations we mentioned above carry relative Frobenius action $\varphi$ induced from the Witt vectors.

- They carry the corresponding Banach or (ind-)Fréchet spaces structures. So we can generalize the corresponding Kedlaya-Liu’s construction to the following situations (here let $G$ be finite):

- We can then consider the corresponding completed Frobenius modules over the rings in the equivariant setting. To be more precise over:

  $$\overline{\nabla}_R[G], \Omega^{\text{int}}_{R,A}, \Omega_{R,A}, \overline{\nabla}_{R,A}, \overline{\nabla}^\text{bd}_{R,A}$$ (0.12)

  one considers the Frobenius modules finite locally free.

- With the corresponding finite locally free models over

  $$\overline{\nabla}'_R[G], \overline{\nabla}'_{R,A}, \overline{\nabla}^\text{bd, r}_{R,A},$$ (0.13)

  again carrying the corresponding semilinear Frobenius structures, where $r$ could be $\infty$.

- One also consider families of Frobenius modules over

  $$\overline{\nabla}'_R[G], \overline{\nabla}'_{R,A},$$ (0.14)

  in glueing fashion with obvious cocycle condition with respect to three intervals $I \subset J \subset K$. These are called the corresponding Frobenius bundles.
One can consider the corresponding schemes attached to the above commutative rings, for instance

\[ \text{Spec}\widetilde{\Pi}_{R,A}, \text{Spec}\widetilde{\Pi}_R[G]. \]  

(0.15)

And consider the corresponding categories:

\[ \text{Mod}(\mathcal{O}_{\text{Spec}\widetilde{\Pi}_{R,A}}), \text{Mod}(\mathcal{O}_{\text{Spec}\widetilde{\Pi}_R[G]}). \]  

(0.16)

These are very straightforward and even crucial especially when we consider

\[ \varphi - \text{Mod}(\mathcal{O}_{\text{Spec}\widetilde{\Pi}_{R,A}}), \varphi - \text{Mod}(\mathcal{O}_{\text{Spec}\widetilde{\Pi}_R[G]}), \]  

(0.17)

in some Frobenius equivariant way.

But on the other hand it is also very convenient to encode the Frobenius action inside the spaces themselves, which leads to Fargues-Fontaine Schemes as those in the work of Kedlaya-Liu.
Deformation of Schemes

- Roughly one takes the corresponding $\varphi = p^n$ equivariant elements in the full Robba ring, and putting them to be a commutative graded ring $\bigoplus P_{R,A,n}$, and then glueing them through the Proj construction by glueing subschemes taking the form of $\text{Spec} P_{R,A}[1/f]_0$.

- Roughly one takes the corresponding $\varphi = p^n$ equivariant elements in the full Robba ring $\tilde{\Pi}_R[G]$, and putting them to be a commutative graded ring $\bigoplus P_{R,G,n}$, and then glueing them through the Proj construction by glueing subschemes taking the form of $\text{Spec} P_{R,G}[1/f]_0$.

- Therefore we have the natural functor:

$$
\text{Mod}\mathcal{O}_{\text{Proj}R,A} \longrightarrow \text{Mod}\mathcal{O}_{\text{Spec}\tilde{\Pi}_R\otimes A},
$$

defined by using the corresponding pullbacks.

- (Theorem, Tong) We have the following categories are equivalent (generalizing the work of Kedlaya-Liu, Kedlaya-Pottharst):

1. The category of all the quasicoherent finite locally free sheaves over $\text{Proj} \bigoplus P_{R,A,n}$;
2. The category of all the Frobenius modules of the global sections of all the $\varphi$-equivariant quasicoherent finite locally free sheaves over $\text{Spec}\tilde{\Pi}_R\otimes A$;
3. The category of all the Frobenius modules over $\tilde{\Pi}_R\otimes A$;
4. The category of all the Frobenius bundles over $\tilde{\Pi}_R.A$.

- For the rings for general $G$, we expect one should also be able to establish some results parallel to this once the structures are more literally investigated. We are also interested in the noncommutative coefficients as in Zähringer’s thesis, but we need to use noncommutative topos.
(Theorem, Tong) We have the following categories are equivalent (generalizing the work of Kedlaya-Liu, Kedlaya-Pottharst):

I. The category of pro-systems of all the quasicoherent finite locally free sheaves over $\text{Proj} \bigoplus P_{R,A_\infty,n}$;

II. The category of pro-systems of all the Frobenius modules coming from the global sections of all the $\varphi$-equivariant quasicoherent finite locally free sheaves over $\text{Spec} \tilde{\Pi}_{R,A_\infty}$;

III. The category of pro-systems of all the Frobenius modules over $\tilde{\Pi}_{R,A_\infty}$;

IV. The category of pro-systems of all the Frobenius bundles over $\tilde{\Pi}_{R,A_\infty}$.

Here $A_\infty$ is a Fréchet-Stein algebra attached to a compact $p$-adic Lie group such that the algebra is limit of (commutative) reduced affinoid algebras. And the finiteness is put on the infinite level of ind-scheme, actually one can also just put on each level.
Deformation of Schemes

- **(Outline)** Following Kedlaya-Liu:
  1. Construct the glueing process over the scheme $\text{Spec}\tilde{\Pi}_{R,A}^\infty$;
  2. The functors could be read off from the corresponding diagram above, namely one glues the resulting sheaves over each $\text{Spec}\tilde{\Pi}_{R,A}^\infty[1/f]$ for each suitable element $f$ in the graded ring, then takes the corresponding global section;
  3. Then from the last category back to the quasicoherent sheaves over the Fargues-Fontaine scheme we need to solve some Frobenius algebraic equation by $p$-adic analytic method to show that taking Frobenius invariance over each affine subspace is exact, where one uses Kedlaya-Liu’s approach which could be dated back to Kedlaya’s approach to slope filtration over extended Robba rings.

- Let us look back the functor:

  $$\text{Mod}\mathcal{O}_{\text{Proj} P_{R,A}} \longrightarrow \varphi - \text{Mod}\mathcal{O}_{\text{Spec}\tilde{\Pi}_{R,A}^\infty} \longrightarrow \varphi - \text{Mod}\mathcal{O}_{\text{Spec}\Pi_{R,A}^\infty},$$

  obviously one might want to generalize the picture above, which was also considered by Kedlaya-Liu in their original work.

- **(Theorem, Tong)** We have the following categories are equivalent (generalizing the work of Kedlaya-Liu, Kedlaya-Pottharst):
  I. The category of all the pseudocoherent sheaves over $\text{Proj} \bigoplus P_{R,A,n}$;
  II. The category of all the pseudocoherent $\varphi$-equivariant modules over $\tilde{\Pi}_{R,A}$. 
Based on the study we did above, it should be very natural to consider more general pseudocoherent complexes in some higher categorical sense. Note that pseudocoherent objects were naturally emerging in SGA from some K-theoretic point of view. Also more importantly Hodge-Iwasawa theory to some extent will behave better if we forget the derived category, when we would like to study the K-theoretic aspects.

**Definition** Let \( Ch\text{Mod}\mathcal{O}_{\text{Proj}P_R} \) denote the category of all the complexes of objects in \( \text{Mod}\mathcal{O}_{\text{Proj}R} \).

**Definition** We now use the notations:

\[
D_{\text{perf}\text{Proj}P_R}, D_{\text{pseudoo}\text{Proj}P_R}
\]

(0.18) to denote the category of all the perfect and pseudocoherent complexes.

**Definition** One also has the following subcategories:

\[
D_{\text{perf}\text{Proj}P_R}^{\text{dg-flat}},
\]

(0.19)

\[
D_{\text{perf}\text{Proj}P_R}^{\text{str}},
\]

(0.20)

**Proposition, after Thomason-Trobaugh** These categories admit Waldhausen structure.

**Question** In the situation where \( R = \tilde{R}_\psi \) attached to the cyclotomic tower, we would like to know if \( D_{\text{perf}\text{Proj}P_R} \) and \( D_{\text{perf}\text{Proj}P_R}^{\text{str}} \) admit Waldhausen exact functors to \( D_{\text{perf}}(\mathbb{Q}_p) \) or \( D_{\text{perf}}^{\text{str}}(\mathbb{Q}_p) \), which induce maps on the associated K-theory spaces.
Comments on Analytic Hodge-Iwasawa Modules

- Now we consider the analytic version of the space above.
- Recall Kedlaya-Liu’s construction of the adic Fargues-Fontaine space which gives rise to a quotient (by using powers of the Frobenius operator) $X_R$ of the space
  \[
  Y_R := \bigcup_{0 < s < r} \text{Spa}(\tilde{\mathcal{O}}_{X_R}^{[s,r]}, \tilde{\mathcal{O}}_{X_R}^{[s,r]} + ).
  \] (0.21)
- This is a locally ringed space $(X_R, \mathcal{O}_{X_R})$, so one can consider the Grothendieck abelian category $\text{Mod}(\mathcal{O}_{X_R})$ which is the category of all the $\mathcal{O}_{X_R}$-sheaves of modules over $X_R$. We use the notation $\text{Mod}^\infty(\mathcal{O}_{X_R})$ to denote the corresponding stable $\infty$-category associated with homotopy category $\text{HMod}^\infty(\mathcal{O}_{X_R})$ as in Lurie’s book. We have the parallel categories for $Y_R$, namely $\varphi \text{Mod}(\mathcal{O}_{Y_R})$ and so on.
- **(Definition)** We have the corresponding sub Grothendieck abelian category $\text{IndMod}^{\text{pseudo}}(\mathcal{O}_{X_R})$ by using results of Kedlaya-Liu (for nice $R$). We use the notation $D^\infty \text{IndMod}^{\text{pseudo}}(\mathcal{O}_{X_R})$ to denote the stable $\infty$-category attached. We have the ones for $Y_R$ as well.
- In this context one can consider the $K$-theory as in the scheme situation by using the ideas and constructions from Blumberg-Gepner-Tabuada. Moreover we can study the Hodge Theory.
- We expect that one can study among these big categories to find interesting relationships, since this should give us the right understanding of the $p$-adic Hodge theory. The corresponding pseudocoherent version comparison could be expected to be deduced as in Kedlaya-Liu’s work.


