This week:

- Regrades for HW2: Tuesday, Feb 4, 8am-11pm
First step analysis. \( N \) states

\((X_n)_{n=0}^\infty\) is a Markov chain on \(\{0,1,2,\ldots, N\}\) with transition probability matrix, first \(r\) states \(\{0,1,\ldots, r-1\}\) are transient, \(\{r, \ldots, N\}\) are absorbing.

\[
P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}
\]

\[
u_i = U_{ik} = P(\text{absorption in } k \mid X_0 = i) = \sum_{j=0}^N P(\text{absorption in } k \mid X_1 = j, X_0 = i) P_{ij}
\]

\[
= P_{ik} + \sum_{j=r}^{N-1} P_{ij} \cdot U_{jk} + \sum_{j=0}^{r-1} P_{ij} \cdot U_{jk}
\]

\[
\Rightarrow U_{ik} = P_{ik} + \sum_{j=0}^{r-1} P_{ij} \cdot U_{jk} \quad \text{for } i \in \{0, \ldots, r-1\}
\]
Example: 4 states

\[
P = \begin{pmatrix}
0.05 & 0.15 & 0.25 & 0.55 \\
0.1 & 0.2 & 0.3 & 0.4 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[U_{ik} = P(\text{trapped in } k \text{ starting from } i)\]

\[i \in \{0, 1\}, \ k \in \{2, 3\}\]
General absorbing Markov chain

States \{0, 1, \ldots, r-1 \} are transient, states \{r, r+1, \ldots \} are absorbing

\[ T = \min \{ n \geq 0, X_n = r \} \] — absorption time

Compute:

Particular choices of \( g \):
General absorbing Markov chain

\[ w_i = E\left( \sum_{n=0}^{T-1} g(X_n) \mid X_0 = i \right) \]

Derive equation for \( w_i \):

\[ w_i = \]

If \( g(j) = 1 \ \forall j \)

\[ w_i = v_i = E(T \mid X_0 = i) \] ← absorption time

Equation:

\[ w_i = W_{ik} \] ← # of visits to \( k \) before absorption

Equation:
Two-state Markov chain

\[ X_n \in \{0, 1\}, \quad a, b \in (0, 1) \]

\[
P = \begin{pmatrix} 0 & 1 \\ 1-a & a \\ b & 1-b \end{pmatrix}
\]

Fact (no proof):
Markov chains defined by indep. r.v.

Let $\xi_0, \xi_1, \ldots$ are i.i.d., $\xi_i \in \{0, 1, 2, \ldots, 3\}$, $P(\xi_0 = i) = a_i$

- $X_n = \xi_n$. (independent r.v.) $(X_n)_{n \geq 0}$

$$P = \begin{pmatrix} \end{pmatrix}$$
Markov chains defined by indep. r.v. (cont)

Let $\xi_0, \xi_1, \ldots$ are i.i.d., $\xi_i \in \{0, 1, 2, \ldots\}$, $P(\xi_0 = i) = a_i$

- **Successive maxima**

$$P = \left( \begin{array}{c} \vdots \end{array} \right)$$

$$\overline{T} := \min \{n \geq 1 : X_n \geq M\} \quad , \quad \mu = E(\overline{T})$$

$$\mu = \ldots$$
Markov chains defined by indep. r.v. (cont)

Let $\xi_0, \xi_1, \ldots$ are i.i.d., $\xi_0 \in \{0, 1, 2, \ldots \} \quad P(\xi_0 = i) = a_i$

Partial sums

$$P(X_{n+1} = j \mid X_n = i)$$

$$P = \begin{pmatrix} \end{pmatrix}$$
One-dimensional random walk

"Position of a moving particle"

\[ X_n \in \{0, 1, 2, \ldots \} \]

\[ P(X_{n+1} = j | X_n = i) = \begin{cases} 
P & \text{if } j = i + 1 \\
1 - P & \text{if } j = i - 1 \\
0 & \text{otherwise}
\end{cases} \]

Gambler's ruin:

Wins with proba \( p \), loses with \( q = 1 - p \)

\[ u_i = U_0 \]
One-dimensional random walk

Solution:

\[ N \to \infty \]