MATH180B: Introduction to Stochastic Processes I
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Today: Random walks and success runs
Next: PK 3.8

This week:

- Regrades for Midterm 1 Tuesday, Feb 11, 8am-11pm
Markov chains defined by indep. r.v. (cont)

Let $\xi_0, \xi_1,...$ are i.i.d. , $\xi_i \in \{0,1,2,..,3\}$, $P(\xi_0 = i) = a_i$

- **Successive maxima**

  $\Theta_n = \max \{\xi_1,...,\xi_n\}$

  $X_n = \Theta_n$. $(X_n)_{n \geq 0}$ is a Markov chain

  \[ P = \begin{pmatrix}
  A_0 & a_1 & a_2 & \cdots \\
  0 & A_1 & a_2 & \cdots \\
  0 & 0 & A_1 & a_2 & \cdots \\
  \end{pmatrix} \]

  $P_{32} = 0$

  $P_{33} = P(\xi_{n+1} \leq 3) = A_3$

  $P_{34} = P(\xi_{n+1} = 4) = a_4$

  $T = \min \{n \geq 1: X_n = M\}$,

  $\mu = E(T)$

  $\mu = \cdots$
Markov chains defined by indep. r.v. (cont)

Let \( \xi_0, \xi_1, \ldots \) are i.i.d., \( \xi_0 \in \{0,1,2,\ldots,3\} \) \( P(\xi_0 = i) = a_i \)

• Partial sums

\[
P(X_{n+1} = j \mid X_n = i)\]

\[
P = \begin{pmatrix} \_ \_ \_ \\
\_ \_ \_ \\
\_ \_ \_ \\
\end{pmatrix}
\]
One-dimensional random walk

"Position of a moving particle"

\[ X_n \in \{0, 1, 2, \ldots \} \quad P(X_{n+1} = j | X_n = i) = \]{

\[ P = \begin{pmatrix}
\end{pmatrix} \]

Gambler's ruin:

Wins with proba p, loses with q (=1-p)

\[ u_i = 0 \]
One-dimensional random walk

Solution:

If $N \to \infty$
Success runs

Markov chain $(X_n)_{n \geq 0}$, $X_n \in \{0, 1, 2, \ldots\}$

\[ P = \begin{pmatrix}
0 & q_0 & 0 & 0 & 0 \\
p_0 & r_1 & q_1 & 0 & 0 \\
p_1 & r_2 & q_2 & 0 & 0 \\
p_2 & r_3 & q_3 & 0 & 0 \\
p_3 & \vdots & \vdots & \ddots & \vdots \\
p_{n-1} & \vdots & \vdots & \ddots & \vdots \\
p_{n-2} & \vdots & \vdots & \ddots & \vdots \\
p_{n-3} & \vdots & \vdots & \ddots & \vdots \\
p_{n} & \vdots & \vdots & \ddots & \vdots \\
p_{\infty} & \end{pmatrix} \]

Special cases

(a) $p_i = p = 1 - \alpha$, $r_i = 0$, $q_i = \alpha$
Success runs

Markov chain \((X_n)_{n \geq 0}\), \(X_n \in \{0,1,2,\ldots\}\)

\[
P = \begin{pmatrix}
0 & q_0 & 0 & 0 & 0 \\
p_1 & r_1 & q_1 & 0 & 0 \\
p_2 & 0 & r_2 & q_2 & 0 \\
p_3 & 0 & 0 & r_3 & q_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\[p_i + r_i + q_i = 1 \quad \forall i\]

Special cases

\((b)\) \(\{\xi_i\} \text{ i.i.d.}, \ P(\xi_i = k) = ak, \ \xi_i \to \text{ lifetime of a bulb}\)
Example

Exercise 3.5.1. You (player A) play the dice game craps against an opponent (player B). In each round both players bet 1$ each and the winner takes all. At the beginning you have 5$, your opponent 10$. Probability of winning in each round is 0.4927. What is the probability that you lose all your money?
Example

Exercise 3.5.6. You sell a baseball trading card on an auction. Let \( \xi_i \) denote the bids and assume that \( \{ \xi_i \} \) are i.i.d., \( P(\xi_i = k) = 0.01 \cdot (0.99)^k \), \( k = 0, 1, 2, \ldots \).

Suppose you want to sell the card for at least $100.$

Q:
Solving the Gambler's Ruin problem

Recall: Gambler's Ruin

\[ P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & q & p & 0 & 0 & \cdots & 0 \\
2 & 0 & q & p & 0 & \cdots & 0 \\
3 & 0 & 0 & q & p & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
N & 0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix} \]

\[ u_i = P \left( X_n \text{ hits } 0 \text{ before } N \mid X_0 = i \right) \]
Other results obtained by the first step analysis

Mean duration in the Gambler's Ruin problem

\[ \nu_i = E(T \mid X_0 = i) \]