Today: Branching processes

Next: PK 4.1

This week:

- Regrades for Midterm 1 Tuesday, Feb 11, 8am-11pm
Example: game of snakes and ladders

Game:
- start from square 1
- on each step, toss a (fair) coin
- H → one step forward
- T → two steps forward
- hit a square with ladder's bottom → go up
- hit a square with snake's head → go down
- the game ends when you reach square 9

Q: What is the average duration of the game?

Solution: Denote $X_n$ = position after $n$ steps, $X_n$ is a Markov process.

$$T = \min\{n: X_n = 9\} \quad E(T \mid X_0 = 1) - \text{average duration of the game}$$

Denote $u_i = E(T \mid X_0 = i)$, First step analysis

$u_2 = u_3$
$$u_1 = 1 + \frac{1}{2} u_2 + \frac{1}{2} u_3$$
$$u_1 - u_3 = \frac{1}{2} (u_3 - u_1) = 0 \quad u_1 = u_3$$

$u_3 = u_5$
$$u_4 = 1 + \frac{1}{2} u_4$$
$$u_4 = 8$$

$u_6 = u_1$
$$u_3 = 1 + \frac{1}{2} u_1 + \frac{1}{2} u_2$$
$$u_3 = 7$$

$u_8 = u_4$
$$u_4 = 1 + \frac{1}{2} u_3 + \frac{1}{2} u_1$$
$$u_2 = 5$$

$u_{10} = u_4$

$P(\text{hitting 9 before 1} \mid X_0 = 5)$

$u_1 = 7$
Solving the Gambler's Ruin problem

Recall: Gambler's Ruin

- win with proba $p$
- lose with proba $q$

\[
P = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
1 & q & p & 0 & \cdots & 0 \\
2 & 0 & q & p & \cdots & 0 \\
3 & 0 & 0 & q & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
N & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}
\]

\[
u_i = P \left( X_n \text{ hits } 0 \text{ before } N \mid X_0 = i \right)
= \frac{(\frac{q}{p})^i - (\frac{q}{p})^N}{1 - (\frac{q}{p})^N}
\]

Take the differences recursively:

\[
(p+q)u_i = u_{i+1} = pu_{i+1} + q \cdot u_i
\]

\[
(p+q)u_2 = u_3 = p \cdot u_3 + q \cdot u_1
\]

\[
(p+q)u_i = u_{i+1} = p \cdot u_{i+1} + q \cdot u_{i-1}
\]

\[
u_0 = 1
\]
\[
u_N = 0
\]

\[-u_i = \left(\frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{N-1}\right)(u_i - 1)
\]

Sum all:

\[
u_N - u_i = \left(\frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{N-1}\right)(u_i - u_0)
\]
Other results obtained by the first step analysis

Mean duration in the Gambler’s Ruin problem

\[ V_i = E(T | X_0 = i) \]

\[ V_i = 1 + pV_{i+1} + qV_{i-1}, \ i = 1, \ldots, N-1 \]

\[ V_0 = 0 \]

\[ V_N = 0 \]

For \( p = q = \frac{1}{2} \)

\[ V_k = k(N-k) \]
**Branching processes: definition**

Suppose $\xi$ is a r.v. $\xi \in \{0,1,2,\ldots,3\}$, $P(\xi = k) = p_k$

$\xi$ models the number of offsprings of an individual

Suppose $\xi_i^{(n)}$, $i, n \in \{0,1,2,\ldots,3\}$ be i.i.d. r.v.'s, $\xi_i^{(n)} \sim \xi$

$X_n =$ size of the $n$-th generation

$X_0 = 1$

$X_1 = \xi_1^{(1)}$ - first generation

$X_2 = \xi_1^{(2)} + \xi_2^{(2)} + \cdots + \xi_{X_1}^{(2)}$ - 2"nd generation

$X_n = \xi_1^{(n)} + \cdots + \xi_{X_{n-1}}^{(n)}$ - $n$"th generation

Galton-Watson process $\xrightarrow{\text{tree}}$ Markov
Branching processes examples

* Population growth

* Survival of family names

(Assume) family name is inherited by sons only.
common ancestor, $\xi$ - number of sons

$X_n$ # individuals (males) of the $n$-th generation
bearing the name.

Q: What is the probability that the name disappears?

* Chain reaction in nuclear fission

Extinction probability
Branching processes: mean and variance

Recall: \( \xi_i \) i.i.d., \( N \) r.v., \( N \in \{0,1,2,\ldots\} \)

\[
E(\xi_i) = \mu, \quad \text{Var}(\xi_i) = \sigma^2, \quad E(N) = \lambda, \quad \text{Var}(N) = \tau^2
\]

For \( X = \sum_{i=1}^{N} \xi_i \)

\[
E(X) = \mu \lambda, \quad \text{Var}(X) = \sqrt{\sigma^2 + \mu^2 \tau^2}
\]

\[
E(\xi) = \mu, \quad \text{Var}(\xi) = \sigma^2
\]

\[
X_1 = \xi^{(1)}_1, \quad E(X_1) = \mu
\]

\[
X_2 = \sum_{i=1}^{X_1} \xi^{(2)}_i, \quad E(X_2) = \mu \cdot \mu = \mu^2
\]

\[
\vdots
\]

\[
X_n = \sum_{i=1}^{X_{n-1}} \xi^{(n)}_i, \quad E(X_n) = \mu. \quad E(X_{n-1}) = \mu^n
\]