Today: Poisson processes

Next: PK 5.2

This week:

- HW 6 due today 11:59 pm
**Poisson distribution**

If $X \sim \text{Pois}(\lambda)$ with $\lambda > 0$, then

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If $Y \sim \text{Pois}(\mu)$ with $\mu > 0$ independent of $X$, then

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Let $\{\xi_i\}_{i=1}^\infty$ be a family of i.i.d. $\text{Ber}(p)$ independent of $X$, consider the random sum $M = \sum_{i=1}^X \xi_i$, $M \sim \text{Bin}(X, p)$. Then

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**Something to remember**

- Little-o notation: two functions \( f(x), g(x) \), limit \( x \to x_0 \in [-\infty, +\infty] \)
  
  \[ f(x) = o(g(x)) \text{ as } x \to x_0 \quad \text{def.} \quad \iff \]

  \[ f(x) \text{ is asymptotically dominated by } g(x) \]

  E.g. 1) \( f(n) = \frac{1}{n^2}, \ g(n) = \frac{1}{n}, \ \text{limit } n \to \infty \)

  2) \( f(x) = x^2, \ g(x) = x, \ \text{limit } x \to 0 \)
Poisson process (Poisson point process)

Def. A Poisson process of intensity (rate) $\lambda > 0$ is an integer-valued stochastic process $(X_t)_{t \geq 0}$ such that

(i) $(X_t)_{t \geq 0}$ has independent increments:

(ii) increments of $(X_t)_{t \geq 0}$ are Poisson r.v.'s

(iii) $X_0 = 0$

$E(X_t) =$  
$\text{Var}(X_t) =$
Example 1

Police department receives calls according to a Poisson process of rate 3 calls per hour. Denote this process \((X_t)_{t \geq 0}\).

There were 4 calls in the first two hours. What is the probability that there will be no call during the next 30 min?

Remark.
Example 2 (PK, p. 226)

Customers arrive in a store according to a Poisson process of rate $\lambda = 4$ per hour. The store opens at 9:00 am. What is the probability that exactly one customer has arrived by 9:30 am and a total of five have arrived by 11:30 am?
Intensity (rate) parameter and nonhomogeneous processes

Interpretation of $\lambda$:

$$P(X_{t+h} - X_t = 1) =$$

$\Rightarrow$ during a short period of time the probability that an event occurs is proportional to the length of the period of time with proportionality constant $\lambda$.

If the Poisson process is homogeneous, then $\lambda$ is constant.

If in (*) we allow $\lambda = \lambda(t)$ depend on $t$, then we get a nonhomogeneous PP. A nonhomogeneous PP with rate func. $\lambda(t)$ is defined as PP with (ii) replaced by...
Example of a nonhomogeneous PP (PK, p. 227)

Let \((X_t)_{t \geq 0}\) be a PP with rate \(\lambda(t) = \begin{cases} 2t, & 0 \leq t < 1 \\ 2, & 1 \leq t < 2 \\ 4-t, & 2 \leq t \leq 4 \end{cases}\)

Compute \(P(X_2 = 2, X_4 - X_2 = 2)\)

Remark. Suppose \((X_t)_{t \geq 0}\) is a PP with rate \(\lambda(t) > 0\).

Define \(\Lambda(t) = \int_0^t \lambda(u) \, du\) and define a process \(Y_t = X_{\Lambda(t)}\)

(note that \(\Lambda\) is strictly increasing thus invertible). Then
Cox processes

Example.
The Law of Rare Events

Recall: \( \{\xi_i\}_{i=1}^{\infty} \), i.i.d. Ber(\( p \)).

Then \( \sum_{i=1}^{N} \xi_i \sim \text{Bin}(N, p) \), counts # of successes

If \( X \sim \text{Bin}(N, p) \) and \( Np = \mu > 0 \)

\( N \) large : many trials

\( p \) small : low success rate (rare event)

Then