Today: Poisson processes and rare events
Next: PK 5.3

This week:
**Poisson process (Poisson point process)**

*Def.* A Poisson process of intensity (rate) \( \lambda > 0 \) is an integer-valued stochastic process \((X_t)_{t \geq 0}\) such that

(i) \((X_t)_{t \geq 0}\) has independent increments:

for any \( t_0 = 0 < t_1 < t_2 < \cdots < t_n \)

\[ X_{t_1} - X_{t_0}, \ X_{t_2} - X_{t_1}, \ldots, \ X_{t_n} - X_{t_{n-1}} \] are independent

(ii) increments of \((X_t)_{t \geq 0}\) are Poisson r.v.'s

for any \( s \geq 0, t > 0 \)

\[ X_{s+t} - X_s \sim \text{Pois}(\lambda t) \]

\[ P(X_{s+t} - X_s = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \]

(iii) \( X_0 = 0 \)
Intensity (rate) parameter and nonhomogeneous processes

Interpretation of \( \lambda \):

\[
P(X_{t+h} - X_t = 1) = \frac{\lambda h}{1!} e^{-\lambda h} = \lambda h (1-\lambda h + \frac{(\lambda h)^2}{2!} - \ldots )
\]

\( = \lambda h + o(h) \quad (\ast) \)

\( \Rightarrow \) during a short period of time the probability that an event occurs is proportional to the length of the period of time with proportionality constant \( \lambda \).

If the Poisson process is homogeneous, then \( \lambda \) is constant.

If in (\ast) we allow \( \lambda = \lambda(t) \) depend on \( t \), then we get a nonhomogeneous PP. A nonhomogeneous PP with rate func. \( \lambda(t) \) is defined as PP with (ii) replaced by

\((ii)'\) for any \( s \geq 0, t > 0 \) \( X_{s+t} - X_s \sim \text{Pois} \left( \int_s^{s+t} \lambda(u) \, du \right) \)
Example of a nonhomogeneous PP (PK, p. 227)

Let \((X_t)_{t \geq 0}\) be a PP with rate \(\lambda(t) = \begin{cases} 
2t, & 0 \leq t < 1 \\
2, & 1 \leq t < 2 \\
4-t, & 2 \leq t \leq 4
\end{cases}\)

Compute \(P(\ X_2 = 2, \ X_4 - X_2 = 2 \) = P(\ X_2 = 2, \ X_4 = 4 \)

\[X_2 \sim \text{Pois}(\int_0^2 \lambda(u) \, du)\ , \ X_4 - X_2 \sim \text{Pois}(\int_2^4 \lambda(u) \, du)\]

\[\int_0^2 \lambda(t) \, dt = 3, \quad P(X_2 = 2) = \frac{3^2}{2!} e^{-3}\]

\[\int_2^4 \lambda(t) \, dt = 2, \quad P(X_4 - X_2 = 2) = \frac{2^2}{2!} e^{-2}\]

\[P(X_2 = 2, \ X_4 - X_2 = 2) = P(X_2 = 2) \cdot P(X_4 - X_2 = 2) = \frac{3^2}{2!} e^{-3} \cdot \frac{2^2}{2!} e^{-2}\]
Time change of non homogeneous PP

Remark. Suppose \((X_t)_{t \geq 0}\) is a PP with rate \(\lambda(t) > 0\) on \((0, +\infty)\)

Define \(\Lambda(t) = \int_0^t \lambda(u) du\) and define a process \(Y_t = X_{\bar{\Lambda}(t)}\)
(note that \(\Lambda\) is strictly increasing thus invertible). Then

(i) \(Y_{s+t} - Y_s = X_{\bar{\Lambda}'(s+t)} - X_{\bar{\Lambda}'(s)} = \text{Pois} (\int_0^{\Lambda'(s+t)} \lambda(u) du)\)

(ii) \(\Lambda'(s+t) = y \Leftrightarrow \Lambda(y) = t \Leftrightarrow \int_0^y \lambda(u) du = t \Leftrightarrow \int_0^{\Lambda'(s+t)} \lambda(u) du = t\)

\(= \int_0^{\Lambda'(s)} \lambda(u) du = (s+t) - s = t\)

(i) & (ii) \(\Rightarrow Y_{s+t} - Y_s \sim \text{Pois}(t)\) for any \(t > 0\)

\(\Rightarrow (Y_t)_{t \geq 0}\) is a (homogeneous) PP with rate \(\lambda = 1\)
Cox processes

Nonhomogeneous PP with rate \( \lambda(t) \),

where \( \lambda(t) \) is a stochastic process, is called Cox process.

Example. Mixed Poisson process.

Let \((X_t)_{t \geq 0}\) be a homogeneous PP with rate 1,

Let \( \Theta \) be a r.v., \( \Theta > 0 \).

Define \((Y_t)_{t \geq 0}\) with \( Y_t = X_{\Theta t} \). \((Y_t)_{t \geq 0}\) is PP with rate \( \Theta \).

If \( \Theta \) is a continuous r.v. with p.d.f. \( f(\Theta) \), then

\[
P(Y_t = k) = \int_0^\infty \frac{(\Theta)^k e^{-\Theta t}}{k!} f(\Theta) d\Theta
\]
The Law of Rare Events

Recall: Let \( \{\xi_i\}_{i=1}^{\infty} \) be i.i.d. \( \text{Ber}(p) \), success/failure

Then \( S_n := \sum_{i=1}^{n} \xi_i \sim \text{Bin}(n, p) \) counts # of successes

If \( np = \mu > 0 \), \( n \) large (many trials), \( p \) small (rare events)

then \( P(S_n = k) \approx \frac{\mu^k}{k!} e^{-\mu} \)

Generalization:

Thm (PK thm. 5.3). Let \( \{\xi_i\}_{i=1}^{\infty} \) be independent r.v.'s, \( \xi_i \sim \text{B}(p_i) \), \( p_i > 0 \)

Denote \( S_n = \sum_{i=1}^{n} \xi_i \) (# of successes), \( \mu = \sum_{i=1}^{n} p_i > 0 \).

Then \( \left| P(S_n = k) - \frac{\mu^k}{k!} e^{-\mu} \right| \leq \sum_{i=1}^{n} p_i^2 \)

Remark. Interesting if \( \sum_{i=1}^{n} p_i^2 \to 0 \) as \( n \to \infty \)
Characterization of the Poisson process

Experiment: count events occurring along \([0, +\infty)\) (time or 1-D space)

Denote by \(N((a,b])\) the number of events that occur on \((a,b]\).

Assumptions:

1. Number of events happening in disjoint intervals are independent

2. For any \(t \geq 0\) and \(h > 0\), the distribution of \(N((t, t+h])\) does not depend on \(t\) (only on \(h\), the length of the interval)

3. There exists \(\lambda > 0\), s.t. \(P(N((t, t+h]) \geq 1) = \lambda h + o(h)\) as \(h \to 0\) (rare events)

4. Simultaneous events are not possible \(P(N((t, t+h]) \geq 2) = o(h)\) as \(h \to 0\)

Then \(X_t = N((0,t])\), then \(X_t\) is a Poisson process with rate \(\lambda\).
Proof that  \( \mathcal{N}(0, t] \sim \text{Pois}(\lambda t) \)

\[
\varepsilon_i = \begin{cases} 
1, & \text{if an event (at least one) occurs on } I_i^{(n)} \\
0, & \text{otherwise}
\end{cases}
\]

(i) split \((0, t]\) into \(n\) smaller disjoint intervals \(I_i^{(n)}, i \in \{1, \ldots, n\}\)

(ii) by 1. and 2. \(\{\varepsilon_i\}\) are i.i.d.

(iii) by 3. \(p_i = P(\varepsilon_i = 1) = \lambda \frac{t}{n} + o\left(\frac{t}{n}\right)\), \(p_i^2 = o\left(\frac{1}{n}\right)\)

(iv) define \(S_n = \sum_{i=1}^{n} \varepsilon_i\) and \(\mu = \sum_{i=1}^{n} p_i = \lambda t + o(1), n \to \infty\)

and apply thm 5.3. \(\left| P(S_n = k) - \frac{\mu^k e^{-\mu}}{k!} \right| \leq o\left(\frac{1}{n}\right)\) as \(n \to \infty\)

true for any \(n\), take \(n \to \infty\)

\(\Rightarrow P(S_n = k) \to \frac{(\lambda t)^k}{k!} e^{-\lambda t} \)