Today: Poisson processes and rare events
Next: PK 5.3
Poisson process (Poisson point process)

Def. A Poisson process of intensity (rate) \( \lambda > 0 \) is an integer-valued stochastic process \((X_t)_{t \geq 0}\) such that

(i) \((X_t)_{t \geq 0}\) has independent increments:

for any \( t_0 = 0 < t_1 < t_2 < \cdots < t_n \)

\[ X_{t_1} - X_{t_0}, \, X_{t_2} - X_{t_1}, \ldots, \, X_{t_n} - X_{t_{n-1}} \]

are independent.

(ii) Increments of \((X_t)_{t \geq 0}\) are Poisson r.v.'s

for any \( s \geq 0, \, t > 0 \)

\[ X_{s+t} - X_s \sim \text{Pois}(\lambda t) \]

\[ P( X_{s+t} - X_s = k ) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \]

(iii) \( X_0 = 0 \)
Interpretation of $\lambda$:

$$P(X_{t+h} - X_t = 1) = \frac{\lambda h}{1!} e^{-\lambda h} = \lambda h \left(1 - \lambda h + \frac{(\lambda h)^2}{2!} - \cdots\right)$$

$$= \lambda h + o(h) \quad (\ast)$$

$\Rightarrow$ during a short period of time the probability that an event occurs is proportional to the length of the period of time with proportionality constant $\lambda$.

If the Poisson process is homogeneous, then $\lambda$ is constant.

If in $(\ast)$ we allow $\lambda = \lambda(t)$ depend on $t$, then we get a nonhomogeneous PP. A nonhomogeneous PP with rate func. $\lambda(t)$ is defined as PP with (ii) replaced by

$$(\text{ii}') \text{ for any } s \geq 0, t > 0, X_{s+t} - X_s \sim \text{Pois} \left( \int_s^{s+t} \lambda(u) \, du \right)$$
Example of a nonhomogeneous PP (PK, p. 227)

Let \((X_t)_{t \geq 0}\) be a PP with rate \(\lambda(t) = \begin{cases} 2t, & 0 \leq t < 1 \\ 2, & 1 \leq t < 2 \\ 4-t, & 2 \leq t \leq 4 \end{cases}\)

Compute \(P(X_2 = 2, X_4 - X_2 = 2)\).

\[
\begin{align*}
\int_0^2 \lambda(t) \, dt &= \quad , \quad P(X_2 = 2) = \\
\int_2^4 \lambda(t) \, dt &= \quad , \quad P(X_4 - X_2 = 2) = \\
P(X_2 = 2, X_4 - X_2 = 2) &= \quad 
\end{align*}
\]
Time change of non-homogeneous PP

Remark. Suppose \((X_t)_{t \geq 0}\) is a PP with rate \(\lambda(t) > 0\) on \((0, +\infty)\). Define

\[ \Lambda(t) = \int_0^t \lambda(u) \, du \]

and define a process \(Y_t = X_{\Lambda(t)}\) (note that \(\Lambda\) is strictly increasing thus invertible). Then
Nonhomogeneous PP with rate $\lambda(t)$,

where

Example. Mixed Poisson process.

Let $(X_t)_{t \geq 0}$ be a homogeneous PP with rate 1,

Let $\Theta$ be a r.v., $\Theta > 0$. 

Cox processes
The Law of Rare Events

Recall: Let \( \{ \xi_i \}_{i=1}^{\infty} \) be i.i.d. \( \text{Ber}(p) \), success/failure

Then \( S_n := \sum_{i=1}^{n} \xi_i \sim \text{Bin}(n, p) \) counts \# of successes

If \( np = \mu > 0 \), \( n \) large (many trials), \( p \) small (rare events)

then \( P(S_n = k) \approx \frac{\mu^k}{k!} e^{-\mu} \)

Generalization:

Thm (PK thm. 5.3).

Remark.
Characterization of the Poisson process

Experiment: count events occurring along \([0, +\infty)\) \(1\)-D space

\[
\begin{array}{c}
0 \quad \times \quad \times \quad \times \quad \times \quad \times \\
\end{array}
\]
\(t\)

Denote by \(N((a,b])\) the number of events that occur on \((a,b]\).

Assumptions:

1. Number of events happening in disjoint intervals are independent

2. For any \(t \geq 0\) and \(h > 0\), the distribution of \(N((t, t+h])\) does not depend on \(t\) (only on \(h\), the length of the interval)

3. There exists \(\lambda > 0\), s.t. \(P(N((t, t+h]) \geq 1) = \lambda h + o(h)\) as \(h \to 0\) (rare events)

4. Simultaneous events are not possible \(P(N((t, t+h]) \geq 2) = o(h)\) as \(h \to 0\)

Then
Proof that $N((0, t]) \sim \text{Pois}(\lambda t)$
Proof that $N((0,t]) \sim \text{Pois}(\lambda t)$ (last step)

$S_n = \# \text{ of intervals containing at least one event}$

$S_n \neq N((0,t])$ iff there is an interval $I_i^{(n)}$ that contains more than one event

Denote $X_t := N((0,t])$. Then split $\{X_t=k\}$ as
Proof of thm 5.3

Thm (PK thm. 5.3). Let \( \{\xi_i\}_{i=1}^\infty \) be independent r.v.'s, \( \xi_i \sim \text{Ber}(p_i) \), \( p_i > 0 \).

Denote \( S_n := \sum_{i=1}^n \xi_i \) (\# of successes) and \( \mu = \sum_{i=1}^n p_i > 0 \).

Then \[ \left| P(S_n = k) - \frac{\mu^k}{k!} e^{-\mu} \right| \leq \sum_{i=1}^n p_i. \]

Coupling technique: in order to prove something about the (marginal) distributions of r.v.'s \( X \) and \( Y \) construct r.v.'s \( X', Y' \) s.t.

(i) \( X' \) has the same (marginal) distribution as \( X \)

(ii) \( Y' \) has the same (marginal) distribution as \( Y \)

(iii) some information is available about the joint distribution of \( (X', Y') \).

Proof.
Proof of thm 5.3

Let \( \{U_i\} \) be i.i.d. r.v's, \( U_i \sim \text{Unif}[0, 1] \).

Define

\[
\varepsilon_i' =
\]

and

\[
X_i' =
\]
Proof of thm 5.3

\[ 1 - \pi \leq e^{-\pi} \]