MATH180B: Introduction to Stochastic Processes I
www.math.ucsd.edu/~ynemish/180b

Today: Probability review (cont.)

Next: PK 2.1, 2.3

This week:

- visit course web site (lecture notes)
- join Piazza
- new office hours on Wednesday
**Gamma distribution**

**Def.** Nonnegative r.v. $X$ has gamma distribution with parameters $\alpha$ and $\lambda$, $X \sim \Gamma(\alpha, \lambda)$, if $f_X(x) =$

\begin{align*}
\text{Remarks: } \int_0^\infty x^{\alpha-1}e^{-\lambda x} \, dx = 
\end{align*}

- For integer $\alpha$, $\Gamma(\alpha, \lambda)$ is the distribution of a sum of $\alpha$ independent r.v.'s having exponential distribution with param. $\lambda$
- Also, generalizes the chi-squared distribution
Joint normal distribution

Distribution of a random vector \((X_1, X_2)\)

**Def. 1:** Let \(\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathbb{R}^2\) and let \(\Sigma \in \mathbb{R}^{2 \times 2}\) be a 2x2 positive-(semi)definite matrix, \(\Sigma > 0\). We say that \((X_1, X_2)\) has a joint normal (bivariate normal) distribution with mean \(\mu\) and covariance matrix \(\Sigma\), denoted equal to

\[
f_{X_1, X_2}(x_1, x_2) = \]

Special case: \(\mu = 0\), \(\Sigma = I \rightarrow\) standard bivariate normal

\[
f_{X_1, X_2}(x_1, x_2) = \]
Joint normal distribution (cont.)

**Proposition 1.** Let $\mu \in \mathbb{R}^2$, let $\Sigma \in \mathbb{R}^{2\times 2}$ be positive definite, let $Y = (Y_1, Y_2)^t$ be independent standard Gaussian r.v.'s. Let $A \in \mathbb{R}^{2\times 2}$ be such that $\Sigma = AA^t$ (always possible).

Then

**Proof.** We will need the following lemma about the density of a linear transformation of a random vector (change of variable in a multivariate integral, no proof).

**Lemma.** Let $U$ be an $n$-dimensional random vector, $B \in \mathbb{R}^{n \times n}$, $\det B \neq 0$, $Z \in \mathbb{R}^n$. 

Joint normal distribution (cont.)
(proof of the proposition).

Using the Lemma, the joint density of $AY + \bar{Y}$ is equal to

**Corollary 1.** Let $X \sim N(\bar{\mu}, \Sigma), \Sigma = AA^T$. Then

**Corollary 2.** Any linear combination of Gaussian random vectors is again a Gaussian random vector.
Joint normal distribution (cont.)

**Corollary 3** If $X = (X_1, X_2) \sim N(\mu, \Sigma)$, then

**Proof.** Compute $E((X - \mu)(X - \mu)^t) = \Sigma$.

Let $Y_1, Y_2$ be independent $N(0, 1)$, denote $Y = (Y_1, Y_2)$, and let $A$ be such that $\Sigma = AA^t$.

Then
Joint normal distribution (cont.)

Compute the density explicitly: let \( \mathbf{\mu} = (\mu_1, \mu_2) \in \mathbb{R}^2 \), \( \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} > 0 \), let \( X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \mathcal{N}(\mathbf{\mu}, \Sigma) \).

\[
f_X(x) = \frac{1}{2\pi \sqrt{\det \Sigma}} \ e^{-\frac{(x-\mathbf{\mu})^t \Sigma^{-1} (x-\mathbf{\mu})}{2}}
\]

\[\det \Sigma =
\]

Then, \( \sqrt{\det \Sigma} =
\]

Similarly, \( \Sigma^{-1} =
\]

\[
(x-\mathbf{\mu})^t \Sigma^{-1} (x-\mathbf{\mu}) =
\]
Joint normal distribution (cont.)

Remarks 1) Definition 1, Proposition 1 and its corollaries 1-3 are not restricted to random vectors of dimension 2; we can generalize the definition by taking 
\[ \mu \in \mathbb{R}^k, \Sigma \in \mathbb{R}^{k \times k}, \Sigma > 0, \] to define a random Gaussian vector \( X \in \mathbb{R}^k, X \sim N(\mu, \Sigma) \) with the joint p.d.f.

\[
    f_X(x) = \frac{1}{(2\pi)^{k/2} |\Sigma|^{1/2}} \exp \left( -\frac{1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu) \right)
\]

Proposition 1 and Corollaries 1-3 remain unchanged.

2) The joint normal distribution is uniquely identified by the

3) If normal random variables are uncorrelated (\( \Sigma = I \)), then