Today: Conditional distribution / random sums

Next: PK

This week:

- HW1 due Friday, January 17, 23:59 pm
- Hint for problem 5: \[
\begin{pmatrix}
\sigma_x^2 & \rho \sigma_x \sigma_z \\
\rho \sigma_x \sigma_z & \sigma_z^2
\end{pmatrix} = A A^t, 
A = \begin{pmatrix}
\sigma_x & 0 \\
x & y \sigma_z
\end{pmatrix}
\]
(find x and y)
Conditional distribution (discrete case)

Recall: for two events $A, B \in \mathcal{F}$, conditional probability of $A$ given $B$ is computed via

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$ 

**Def.** Let $X, Y$ be two discrete r.v.'s taking values in $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ correspondingly. The conditional probability mass function of $X$ given $Y$ is defined by

$$P(X = x_i \mid Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}.$$ 

By the law of total probability ($\{Y = y_j\}_{j=1}^{\infty}$ is a partition).
Conditional joint distribution

Notation: for r.v. $X$ taking values in $\{x_1, x_2, \ldots, x_n\}$ and $f: \mathbb{R} \to \mathbb{R}$, the textbook uses notation $\sum_{i=1}^{n} f(x_i) =: \sum_{x} f(x)$, which may cause confusion.

Def (joint cond. distribution)

Let $X, Y, Z$ be r.v.'s. Then the conditional joint distribution of $(X, Z)$ given $Y = y$ is defined by

$$P_{X, Z | Y}(x_i, z_k | y_j) = \ldots$$

Remark. For any fixed $y_j$, any conditional (joint) distribution (of $X$ or $(X, Z)$ or...) given $Y = y_j$ is itself a (joint) probability distribution.
Example

Let $\mathbf{M} \sim \text{N, } p, q \in (0, 1)$

Let $N \sim \text{B}(M, q)$ and let $X \sim \text{B}(N, p)$. In other words, for $n \in \{0, 1, \ldots, M\}$

$\Pr(X | N) = \binom{N}{k} p^k (1-p)^{N-k}$

What is the (marginal) distribution of $X$? $\Pr_X(k) - ?$

By the law of total probability, for $k \in \{0, 1, \ldots, M\}$
**Conditional expectation (discrete case)**

**Definition.** Let $X, Y$ be discrete r.v.'s with values $\{x_i\}$, $\{y_j\}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $|E(g(X))| < \infty$.

The conditional expectation of $g(X)$ given $Y = y_j$ is defined by

$$E(g(X) | Y = y_j).$$

Similarly for $E(g(X, Z) | Y = y_j)$.

**Remark.** $E(g(X) | Y)$

By the law of total probability
Properties of the conditional expectations

Let $X, Y, Z$ be r.v.'s defined on the same probability space. Let $g : \mathbb{R} \to \mathbb{R}$ and $V : \mathbb{R}^2 \to \mathbb{R}$ be s.t. $E(|g(X)|) < \infty$, $E(|V(X,Y)|) < \infty$.

Recall, for fixed $y_j$, the (joint) distribution of $X$ (or $(X, Z)$) given $Y = y_j$ is a (joint) probability distribution. Conditional expectations have the following properties:

(some analogous to the properties of usual expectation)

1. (Linearity) $E\left(c \cdot g_1(X) + c_2 g_2(Z) \mid Y = y_j\right)$

2. If $g \geq 0$, then $E(g(X) \mid Y = y_j)$

3. $E(V(X,Y) \mid Y = y_j)$
Properties of the conditional expectation

4. $E(g(x) | Y = y_j) =$ if $X$ and $Y$ are independent

5. $E(g(x) h(Y) | Y = y_j)$

6. $E(g(x) h(Y))$

Proof:
Conditional expectation (cont.)

We can also define for a r.v. $X$ and any event $A \in \mathcal{F}$

$$\begin{align*}
    p_X(x | A) &= \frac{P(\{X=x\} \cap A)}{P(A)},
    E(X | A) = \sum_{i=1}^{\infty} x_i p_X(x_i | A)
\end{align*}$$

Example (Exercise 2.1.5)

Let $X \sim \text{Pois}(\lambda)$ and let $A = \{X \text{ is odd} \}$. 