Today: Random sums / Markov chains

Next: PK 3.4

This week:

- HW2 due Friday, January 24, 23:59 pm
Last time

Mean of a random sum:

\[ E\left( \sum_{i=1}^{N} \xi_i \right) = E(\xi_1)E(N) \]

Variance of a random sum:

\[ \text{Var}\left( \sum_{i=1}^{N} \xi_i \right) = E(N)\text{Var}(\xi_1) + \text{Var}(N)(E(\xi_1))^2 \]

Example (number of offsprings of the second generation).

Each parent produces \( k \) offsprings with probability \( p(k) \).
Distribution of random sums: n-fold convolution

**Def.** Let \( f(x) \) be a density of a r.v., let \( n \in \mathbb{N} \).

The n-fold convolution of \( f \) is defined recursively by

\[
X := \sum_{i=1}^{n} \xi_i \quad \text{has p.d.f.}
\]

For r.v. \( N \in \{1, 2, \ldots \} \), \( X | N=n \) has (conditional) p.d.f. \( f^{(n)}(x) \), and by the law of total probability

\[
f_X(x) = \sum_{n=1}^{\infty} f^{(n)}(x) P_N(n)
\]
Example: Geometric sum of Exponential r.v.'s

Let $N \sim \text{Geom}(p)$, let $\xi_i$, $i \in \mathbb{N}$, be i.i.d., $\xi_i \sim \text{Exp}(\lambda)$, $p \in (0,1)$, $\lambda > 0$. Define $X = \sum_{i=1}^{N} \xi_i$. What is the distr. of $X$?

For fixed $n$, we have seen that $\sum_{i=1}^{n} \xi_i \sim$ so $f^{(n)}(x) =$

Therefore, $f_X(x) =$
Example: Stock price changes

Model the changes in the price of a single share of some stock (at the end) of two consecutive trading days.

\[ X_1 - X_0 = Z = \sum_{i=0}^{n} \xi_i \], where \( \xi_i \) is the change from \( i \)-th transaction

\( \xi_0 \) - overnight change

Assumptions: \( \xi_i \) are i.i.d., \( \xi_i \sim N(0, \sigma^2) \)

1) if \( n \) is large, then \( Z \sim N(0, (n+1)\sigma^2) \)

2) if \( Z = \sum_{i=1}^{N} \xi_i \), \( \xi_i \) and \( N \) are indep., \( N \sim \text{Pois}(\lambda) \), then

for any \( m \geq 0 \)

\[ f_\mathbb{Z|N}(x|m) = \frac{f_\mathbb{Z}(x) = \sum_{m=0}^{8} f_\mathbb{Z|N}(x|m) p_N(m)}{\lambda = n} \]
**Markov chains. Introduction**

\{X_n\}_{n=0}^\infty \text{ discrete time random process, } X_n \in \{0,1,2,\ldots\}

**Characteristic Markov property:**

the process retains no memory of where it has been in the past; only the current state of the process can influence where it goes next.

- many phenomena in nature have Markov property
- Markov processes are nice to work with (linear algebra)

**Def.** Let \{X_n\}_{n=0}^\infty be a discrete time stochastic process taking values in \mathbb{Z}_+=\{0,1,2,\ldots\} (for convenience), \{X_n\}_{n=0}^\infty is called Markov chain, if for any \(n \in \mathbb{N}\), and \(i_0, i_1, \ldots, i_{n-1}, i, j \in \mathbb{Z}_+\)
Markov chains. Example 1

Markov chain described by a diagram

Questions:
(a) Starting from 0, what is the probability of hitting 6?
(b) Starting from 1, what is the probability of hitting 3?
(c) Starting from 1, how many steps (on average) to hit 3?
(d) Starting from 1, what is the long-run proportion of time spent in 2?
**One-step transition probability**

**Def.** \( P_{ij}^{n,n+1} := P(X_{n+1} = j \mid X_n = i) \)

is called the one-step transition probability.

(probability to move from state \( i \) to state \( j \))

\( P_{ij}^{n,n+1} = P_{ij} \) for all \( n \to \) stationary transition probabilities

\( P := (P_{ij})_{i,j=0}^\infty \to \) transition probability matrix

**Example 1.**

\[
P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

**Properties**

\( P_{ij} \geq 0 \), for any \( i \in \{0,1,\ldots,3\} \)

\( \sum_{j=0}^8 P_{ij} = 1 \)
Finite dimensional probabilities

Fact. For any stochastic process \((X_t)_{t \in T}\) (discrete or continuous time), a lot of information is contained in the (collection of) finite-dimensional probabilities.

Proposition. Let \(\{X_n\}_{n=1}^\infty\) be a Markov chain with stationary transition probabilities. Let \(P = (P_{ij})_{i,j=1}^\infty\) be the transition probability matrix and let \(p_k = P(X_0 = k)\) be the initial distribution. Then the finite-dimensional probabilities \(P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)\) are completely defined by \(P\) and \(p := (p_k)_{k=0}^\infty\).
Proof of Proposition

Fix $n, \ i_0, \ldots, i_n \in \mathbb{Z}_+$,

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) =$$

Remark

$$P(X_{n+1} = j_{n+1}, X_{n+2} = j_{n+2}, \ldots, X_{n+m} = j_{n+m} | X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$

$$= P(X_{n+1} = j_{n+1}, X_{n+2} = j_{n+2}, \ldots, X_{n+m} = j_{n+m} | X_n = i_n)$$
Theorem.

(probability to hit j from i in n steps is given by)

Proof. Define $P_{i,j}^{(n)} = P(X_{m+n} = j \mid X_m = i)$. Then

$$P_{i,j}^{(n)} = \frac{P(X_{m+n} = j, X_m = i)}{P(X_m = i)} =$$
Example: the Ehrenfest model

Remark. If the initial distribution is given by \( p = (p_i)_{i=0}^{\infty} \), then the distribution after \( n \) steps is given by

\[ \text{(the vector)} \quad P\pi \quad (\text{i.e., } P(X_n=j) = (P^n\pi)_j) \]

The model: diffusion through a membrane

2a balls (particles) in 2 urns A, B (containers)

At each step you choose (unif. at random) a ball and put it to the other urn (particle passing through the membrane).

Let \( Y_n = \# \text{balls in } A, \quad X_n = Y_n - a, \quad X_n \in \{-a, -a+1, \ldots, 0, 1, \ldots, a\} \)