MATH180B: Introduction to Stochastic Processes I
www.math.ucsd.edu/~ynemish/180b

Today: First step analysis
Next: PK 3.5

This week:

- HW3 due Friday, January 32, 23:59 pm
- Regrades for HW2: Tuesday, Feb 4, 8am-11pm

No Homework
First step analysis. The simplest setting: 3 states

Suppose we have Markov chain on \{0,1,2\} with transition probability matrix

\[
P = \begin{pmatrix}
0 & 1 & 2 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[P_{ii} = \beta\]
\[\alpha + \beta + \gamma = 1\]

Start at \(X_0 = 1\)

If the process leaves 1, it is trapped in 0 or 2

Questions: Where (in which state) will the process be trapped?
How long will the process stay at 1 (on average)?

Both questions lead to simple linear equations with coefficients \(\alpha, \beta, \gamma\).
First step analysis. First solutions

Denote:

\[ T = \min\{ n \geq 0 : x_n = 0 \text{ or } x_n = 2 \} \text{ absorption time} \]
\[ u = P(X_T = 0 \mid X_0 = 1) \text{ (probability to end up at 0 starting from 1)} \]

What happens at \( n = 1 \)? Condition on possible outcomes

\[ P(X_T = 0 \mid X_0 = 1, X_1 = 0) = 1 \]
\[ P(X_T = 0 \mid X_0 = 1, X_1 = 2) = 0 \]
\[ P(X_T = 0 \mid X_0 = 1, X_1 = 1) = u = P(X_T = 0 \mid X_0 = 1) \]

\[ u = P(X_T = 0 \mid X_0 = 1) = \sum_{i=0}^{2} P(X_T = 0 \mid X_0 = 1, X_1 = i) \cdot P(X_1 = i \mid X_0 = 1) \]

\[ = P(X_T = 0 \mid X_0 = 1, X_1 = 0) \cdot P(X_1 = 0 \mid X_0 = 1) + \ldots \]
\[ = 1 \cdot P_{10} + u \cdot P_{11} + 0 \cdot P_{12} = 1 \cdot \lambda + u \cdot \beta + 0 \cdot \gamma \]

\[ \Rightarrow u = \lambda + u \cdot \beta \Rightarrow u = \frac{\lambda}{1 - \beta} = \frac{\lambda}{\lambda + \gamma} \]
Example

Questions:

(a) Starting from 0, what is the probability of hitting 6? \( \frac{1}{4} \)

\[
P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & \frac{R}{1} \\ \frac{1}{5} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}
\]

\[u = \begin{pmatrix} \alpha \\ \frac{1}{5} \\ \frac{1}{3} \end{pmatrix}
\]

\[u = \alpha + \beta u = \frac{1}{5} + \frac{1}{5} u = \frac{1}{5} + \frac{1}{5} \left( \frac{1}{5} + \frac{1}{5} u \right) + \ldots
\]

\[u = \frac{\alpha}{1 - \beta} = \frac{1}{5} \cdot \frac{1}{\frac{1}{5}} = \frac{1}{4}
\]
First step analysis. First solutions

Denote:

\[ T = \min \{ n \geq 0 : x_n = 0 \text{ or } x_n = 2 \text{ } \} \text{ - absorption time (as above)} \]

\[ J = E(T | X_0 = 1) \text{ (time (expected) spent at 1)} \]

Try to get an equation for \( J \):

\[ J = E(T | X_0 = 1) = \sum_{i=0}^{2} E(T | X_0 = 1, X_i = i) P(X_i = i | X_0 = 1) \]

\[ \quad = \sum_{i=0}^{2} E(T | X_0 = 1, X_i = i) \cdot P_{ii} \]

\[ \quad = 1 \cdot 2 + (1 + \nu) \cdot \beta + 1 \cdot \gamma = 1 + \beta \nu \]

\[ \quad J = \frac{1}{1 - \beta} \]

Double check:

\[ P(T = k | X_0 = 1) = \beta^{k-1} \cdot (1 - \beta) \text{ for } k = 1, 2, \ldots \]

\[ E(T | X_0 = 1) = \sum_{k=1}^{\infty} k \beta^{k-1} (1 - \beta) = \frac{1}{1 - \beta} \]
Example \( (+ \text{ more than one step}) \)

Questions:

\[
T = \min \{ n : X_n = 1 \text{ or } X_n = 4 \} \\
E(T \mid X_0 = 0) = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4}
\]

\( (c) \) Starting from 1, how many steps (on average) to hit 3? 3

Let \( T = \min \{ n : X_n = 3 \} \), start from 1

\[
p = E(T \mid X_0 = 1) \\
p = E(T \mid X_0 = 1, X_1 = 2) = E(T \mid X_0 = 1, X_1 = 2, X_2 = 3) \cdot P_{23} + \\
+ E(T \mid X_0 = 1, X_1 = 2, X_2 = 1) \cdot P_{21} = 2 + \frac{2}{5} + (2 + p) \cdot \frac{1}{5} = p = 2 + p \cdot \frac{1}{3}
\]
First step analysis, $N$ states

$(X_n)_{n \geq 0}$ is a Markov chain on $\{0, 1, 2, \ldots, N\}$ with transition probability matrix. First $r$ states $\{0, 1, \ldots, r-1\}$ are transient, $\{r, \ldots, N\}$ are absorbing.

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

$$u_i = U_{ik} = P(\text{absorption in } k \mid X_0 = i)$$

$$= \sum_{j=0}^{N} P(\text{absorption in } k \mid X_0 = i, X_1 = j) \cdot P_{ij}$$

$$u_i = 1 \cdot P_{ik} + 0 \cdot \sum_{j \neq i} \frac{N}{N} P_{ij} + \sum_{j=0}^{r-1} u_j \cdot P_{ij}$$