1. (25 points) Consider a Markov chain whose transition probability matrix is given by

\[
P = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 0.2 & 0.4 & 0.2 & 0.1 & \alpha \\
1 & 0.2 & 0.3 & 0.3 & 0 & 0.2 \\
2 & 0 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]  

(a) (3 points) Determine the value of \( \alpha \).

Solution. Each row of a transition probability matrix must be equal to 1, thus

\[
\alpha = 1 - (0.2 + 0.4 + 0.2 + 0.1) = 0.1.
\]  

(b) (22 points) Starting in state 0, determine the probability that the Markov chain ends in state 3.

Solution. States 3 and 4 are absorbing. Denote by \( T \) the absorption time \( T = \min\{n : X_n \in \{3, 4\}\} \) and denote by \( u_i \) the probability of getting absorbed by state 3 starting from state \( i, i \in \{0, 1, 2\} \), i.e.,

\[
u_i = P(X_T = 3 \mid X_0 = i).
\]  

Applying the first step analysis leads to the following system of equations

\[
\begin{align*}
u_0 &= 0.2u_0 + 0.4u_1 + 0.2u_2 + 0.1, \\
u_1 &= 0.2u_0 + 0.3u_1 + 0.3u_2, \\
u_2 &= u_1.
\end{align*}
\]  

Plugging (6) into (5) gives

\[
u_1 = 0.2u_0 + 0.3u_1 + 0.3u_1 \implies u_1 = 0.5u_0.
\]  

Plugging (6) and (7) into (4) gives

\[
u_0 = 0.2u_0 + 0.2u_0 + 0.1u_0 + 0.1 \implies u_0 = 0.2.
\]  

Thus, \( P(X_T = 3 \mid X_0 = 0) = 0.2 \).
2. (25 points) The number of offspring in a population has a shifted Geometric distribution with parameter $p \in (0, 1)$, i.e., if $\xi$ denotes the number of offspring of a certain individual, then

$$P(\xi = k) = p(1 - p)^k$$

for $k \in \{0, 1, 2, 3, \ldots\}$.

(a) (10 points) Denote by $u_n$ the probability that the population will go extinct on step $n$ or earlier. Express $u_n$ as a function of $u_{n-1}$ (simplify all infinite sums).

**Solution.** Let $(X_n)_{n \geq 0}$ be the Markov process of the population size starting from $X_0 = 1$. Denote the probability of getting extinct before time $n$ by $u_n$

$$u_n = P(X_n = 0).$$

Then

$$u_n = \sum_{k=0}^{\infty} P(\xi = k)(u_{n-1})^k$$

$$= \sum_{k=0}^{\infty} p(1 - p)^k(u_{n-1})^k$$

$$= p \sum_{k=0}^{\infty} ((1 - p)u_{n-1})^k$$

$$= p \frac{1}{1 - (1 - p)u_{n-1}},$$

where at the last step we used that $|(1 - p)u_{n-1}| < 1$ for any $p \in (0, 1)$ and any $n \geq 1$.

(b) (5 points) Denote the extinction probability by $u := \lim_{n \to \infty} u_n$. What quadratic equation must be satisfied by $u$? (Use the result of part (a)).

**Solution.** From part (a) we have that

$$u_n = p \frac{1}{1 - (1 - p)u_{n-1}}.$$

Taking the limit $n \to \infty$ in (15) gives

$$u = p \frac{1}{1 - (1 - p)u} \Rightarrow u(1 - (1 - p)u) = p.$$

Thus $u$ satisfies

$$(1 - p)u^2 - u + p = 0.$$
(c) (10 points) By solving the equation from (b), compute the extinction probability \( u \) for \( p \in [1/2, 1) \).

**Solution.** Denote the solutions to the equation (17) by \( u^{(1)} \) and \( u^{(2)} \). It is clear that one of the solutions to (17) is

\[
u^{(1)} = 1.
\]

(18)

Then

\[
u^{(2)} = \frac{p}{1 - p}.
\]

(19)

If \( p \in [1/2, 1) \), then \( p \geq 1 - p \), so \( \frac{p}{1 - p} \geq 1 \). Since \( u \leq 1 \), we conclude that \( u = 1 \) for \( p \in [1/2, 1) \).

This means that if \( p \in [1/2, 1) \), then the population will get extinct in the long run with probability 1 (almost surely).
3. (25 points) Every day a particular customer buys one yogurt and consumes it. The customer chooses between three brands (denote them 0, 1 and 2), paying $1 + j$ dollars for brand $j$, $j \in \{0, 1, 2\}$. The customer switches brands depending on what brand of yogurt he consumed the day before according to a Markov chain with the following transition probability matrix

$$P = \begin{bmatrix}
0 & 0 & 0 \\
0.6 & 0.4 & 0 \\
0.2 & 0.6 & 0.2
\end{bmatrix} \quad (20)$$

(a) (15 points) What fraction of time is the customer consuming the cheapest brand 0?

**Solution.** Transition probability matrix $P$ is regular: $(P^2)_{ij} > 0$ for any $i, j \in \{0, 1, 2\}$. By the limit theorem for Markov chains with regular transition probability matrix, the limiting distribution is given as a strictly positive solution to the following system of equations

$$\pi_0 = 0.2\pi_1 + 0.4\pi_2, \quad (21)$$
$$\pi_1 = 0.6\pi_0 + 0.6\pi_1 + 0.6\pi_2, \quad (22)$$
$$\pi_2 = 0.4\pi_0 + 0.2\pi_1, \quad (23)$$
$$1 = \pi_0 + \pi_1 + \pi_2. \quad (24)$$

From (22) and (24) we have that

$$\pi_1 = 0.6. \quad (25)$$

By subtracting (21) from (23) we get that

$$\pi_0 = \pi_2. \quad (26)$$

Plugging (25) and (26) into (24) implies

$$\pi_0 = \pi_2 = 0.2. \quad (27)$$

In the long run the limiting distribution gives the fraction of time spent in particular state. Therefore, in the long run 20% of the time the customer consumes the brand 0.

(b) (10 points) In the long run, how much money does the customer spend daily on yogurts?

**Solution.** From part (a) we have that in the long run 20% of the time the customer consumes yogurt 0, 60% of the time yogurt 1 and 20% of the time yogurt 2. Thus of average every day the customer spends

$$0.2 \cdot 1 + 0.6 \cdot 2 + 0.2 \cdot 3 = 2 \quad (28)$$

dollars on yogurts.
4. (25 points) Let \( Y \) be a random variable taking values in the set of all integer numbers \( \{0, \pm 1, \pm 2, \pm 3, \ldots \} \) with probability mass function \( P(Y = k) = \alpha_k > 0 \) for \( k \in \{0, \pm 1, \pm 2, \pm 3, \ldots \} \). Consider a Markov chain \( (X_n)_{n \geq 0} \) on the set \( \{0, \pm 1, \pm 2, \pm 3, \ldots \} \) with stationary one-step transition probability matrix

\[
P = \begin{bmatrix}
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\
\cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-3 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
-2 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
-1 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & \cdots & \alpha_{-3} & \alpha_{-2} & \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\
1 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
2 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
3 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
4 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots 
\end{bmatrix}
\] (29)

You may assume without proof that \( (X_n)_{n \geq 0} \) is irreducible and aperiodic.

(a) (10 points) Draw a graph (diagram) representing the dynamics of \( (X_n)_{n \geq 0} \) on the set \( \{-2, -1, 0, 1, 2\} \). Show that \( (X_n)_{n \geq 0} \) is always recurrent. (Hint. Compute \( f_{00} \). Note that you can return to 0 in \( k \) steps either from the right or from the left.)

Solution. The diagram restricted to the states \( \{-2, -1, 0, 1, 2\} \) has the following form

Markov chain \( (X_n)_{n \geq 0} \) is irreducible, so in order to prove that \( (X_n)_{n \geq 0} \) is recurrent it is enough to show that one of the states is recurrent. Consider state 0. The \( k \)-step first return probability for state 0 is given by

\[
f_{00}^{(k)} = \begin{cases} 
\alpha_0, & k = 1 \\
\alpha_{k-1} + \alpha_{-(k+1)}, & k \geq 2.
\end{cases}
\] (30)

Now

\[
f_{00} = \sum_{k=1}^{\infty} f_{00}^{(k)} = \alpha_0 + (\alpha_1 + \alpha_{-1}) + \alpha_2 + \alpha_{-2} + \cdots = \sum_{\ell=-\infty}^{\infty} \alpha_\ell = 1
\] (31)

since \( \alpha_\ell, \ell \in \{0, \pm 1, \pm 2, \ldots\} \), is the distribution (probability mass function) of the random variable \( Y \). Since \( f_{00} = 1 \), we conclude that state 0 (and all the other states) are recurrent.

(b) (10 points) Show that \( (X_n)_{n \geq 0} \) is positive recurrent if and only if \( E(|Y|) < \infty \). (Recall that \( E(|Y|) = 1(\alpha_1 + \alpha_{-1}) + 2(\alpha_2 + \alpha_{-2}) + 3(\alpha_3 + \alpha_{-3}) + \cdots \))

Solution. The criterion for positive recurrence is that the mean first return time is finite

\[
m_0 = \sum_{k=1}^{\infty} k f_{00}^{(k)} < \infty.
\] (32)
Compute $m_0$:

$$\sum_{k=1}^{\infty} k f_{00}^{(k)} = \alpha_0 + 2(\alpha_1 + \alpha_{-1}) + 3(\alpha_2 + \alpha_{-2}) + 4(\alpha_3 + \alpha_{-3}) + \cdots$$ (33)

$$= \alpha_0 + (\alpha_1 + \alpha_{-1}) + (\alpha_2 + \alpha_{-2}) + (\alpha_3 + \alpha_{-3}) + \cdots$$ (34)

$$+ 1(\alpha_1 + \alpha_{-1}) + 2(\alpha_2 + \alpha_{-2}) + 3(\alpha_3 + \alpha_{-3}) + \cdots$$ (35)

$$= 1 + E(|Y|).$$ (36)

We see that $m_0$ is finite if and only if $E(|Y|) < \infty$.

(c) (5 points) Assume that $E(|Y|) < \infty$. By computing the mean first return time, determine the limit

$$\pi_0 = \lim_{n \to \infty} P_{00}^{(n)}.$$ What is the relation between $\pi_0$ and $E(|Y|)$?

**Solution.** Since $E(|Y|) < \infty$, we conclude from part (b) that the Markov chain $(X_n)_{n \geq 0}$ is positive recurrent. By thy limit theorem for positive recurrent aperiodic Markov chains, there exists the limiting distribution $\pi = (\pi_i)_{i=\infty}^{\infty}$ and

$$\lim_{n \to \infty} P_{ii}^{(n)} = \pi_i = \frac{1}{m_i} > 0.$$ (37)

In particular, for the state 0

$$\lim_{n \to \infty} P_{00}^{(n)} = \pi_0 = \frac{1}{m_0} = \frac{1}{1 + E(|Y|)}.$$ (38)