MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/180c

Today: Introduction. Birth processes
> Q&A: April 1
Next: PK 6.2-6.3

This week:

- visit course web site
- homework #0
- join Piazza
- no discussion sections
Stochastic (random) processes

Def. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Stochastic process is a collection $(X_t : t \in T)$
of random variables $X_t : \Omega \to S \subset \mathbb{R}$ (all defined on the
same probability space)

- often $t$ represents time, but can be 1-D space
- $T$ is called the index set, $S$ is called the state space
- $X : \Omega \times T \to S$ ($X_t(\omega) \in S$)
- for any fixed $\omega$, we get a realization of all
  random variables $(X_t(\omega) : t \in T)$ ← sample path trajectory
  $X_\omega(\omega) : T \to S$ ← function in $\omega$

- stochastic process = random function
Questions:

- What is $T$?
- What is $S$?
- Relations between $X_{t_1}$ and $X_{t_2}$ for $t_1 \neq t_2$?
- Properties of the trajectory?

Discrete time  
$\begin{align*}
T & = \mathbb{N}, \mathbb{Z}, \text{finite set} \\
S & = \mathbb{R} \\
\text{Real-valued} \\
\text{Continus sample path}
\end{align*}$

Continuous time  
$\begin{align*}
T & = \mathbb{R}, [0, +\infty), [0,1] \\
S & = \mathbb{R}, \mathbb{Z}, \mathbb{C} \cap [0, +\infty) \\
\text{Random vector} \\
\text{Nonnegative} \\
\text{Sample path}
\end{align*}$
Examples of stochastic processes

- Gaussian processes: for any $t \in T$, $X_t$ has normal distrib.
- Stationary processes: distribution doesn't change in time
- Processes with stationary / independent increments (Lévy)
- Poisson process: increments are independent and Poisson
- Markov processes: "distribution in the future depends only on the current state, but does not depend on the past"
Examples of stochastic processes

- Martingales: \( E[X_{n+1} | X_n, X_{n-1}, \ldots, X_0] = X_n \) ("fair game")

- Brownian motion (Wiener process) is continuous-time s.p. Gaussian, martingale, has stationary and independent increments, Markov, \( \text{Var}[W_t] = t \), \( \text{Cov}(W_t, W_s) = \min\{s, t\} \), its sample path is everywhere continuous and nowhere differentiable

- Diffusion processes (stochastic differential equations)
Continuous time MC
Continuous Time Markov Chains

Def (Discrete-time Markov chain)
Let \((X_n)_{n \geq 0}\) be a discrete time stochastic process taking values in \(\mathbb{Z}_t = \{0, 1, 2, \ldots\}\) (for convenience). \((X_n)_{n \geq 0}\) is called Markov chain if for any \(n \in \mathbb{N}\) and \(i_0, i_1, \ldots, i_{n-1}, i, j \in \mathbb{Z}_t\),
\[
P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i) = P(X_{n+1} = j \mid X_n = i)
\]

Def (Continuous-time Markov chain)
Let \((X_t)_{t \geq 0} = (X_t : 0 \leq t < \infty)\) be a continuous time process taking values in \(\mathbb{Z}_t\). \((X_t)_{t \geq 0}\) is called Markov chain if for any \(n \in \mathbb{N}\), \(0 \leq t_0 < t_1 < \cdots < t_{n-1} < s, t > 0\), \(i_0, i_1, \ldots, i_{n-1}, i, j \in \mathbb{Z}_t\),
\[
P(X_{s+t} = j \mid X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}, X_s = i) = P(X_{s+t} = j \mid X_s = i)
\]
Markov property $\uparrow$
Example: Poisson process as MC

Is Poisson process a continuous time MC?

Poisson process:

✓ continuous time
✓ discrete state
✓ Markov property

Let \((X_t)_{t \geq 0}\) be a Poisson process, let \(i_0 \leq i_1 \leq \ldots \leq i_{n-1} \leq i \leq j\)

\[
P(X_{s+t} = j \mid X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}, X_s = i) = \frac{P(X_{s+t} - X_s = j - i \mid X_s = i)}{P(\sum_{t_0}^{t_{n-1}} X_{t_{k+1}} - X_{t_k} = i_{k+1} - i_k \mid X_s = i)}
\]

\[
= \frac{P(X_{s+t} - X_s = j - i)}{P(\sum_{t_0}^{t_{n-1}} X_{t_{k+1}} - X_{t_k} = i_{k+1} - i_k)}
\]

\[
= P(X_{s+t} - X_s = j - i \mid X_s = i) = P(X_{s+t} = j \mid X_s = i)
\]
Transition probability function

One way of describing a continuous time MC is by using the transition probability functions.

**Def.** Let \((X_t)_{t \geq 0}\) be a MC. We call

\[
P(X_{s+t} = j \mid X_s = i), \quad i, j \in \{0, 1, \ldots\}, \quad s \geq 0, \quad t > 0
\]

the transition probability function for \((X_t)_{t \geq 0}\).

If \(P(X_{s+t} = j \mid X_s = i)\) does not depend on \(s\), we say that \((X_t)_{t \geq 0}\) has stationary transition probabilities and we define

\[
P_{ij}(t) := P(X_{s+t} = j \mid X_s = i) \quad (= P(X_t = j \mid X_0 = i))
\]

[compare with \(n\)-step transition probabilities]
Characterization of the Poisson process

Experiment: Count events occurring along $[0, +\infty)$ for 1-D space

\[ 0 \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \ldots \quad \text{for time} \quad t \]

Denote by $N((a, b])$ the number of events that occur on $(a, b]$.

Assumptions:

1. Number of events happening in disjoint intervals are independent.
2. For any $t \geq 0$ and $h > 0$, the distribution of $N((t, t+h])$ does not depend on $t$ (only on $h$, the length of the interval).
3. There exists $\lambda > 0$ s.t. $P(N((t, t+h]) \geq 1) = \lambda h + o(h)$ as $h \to 0$ (rare events)
4. Simultaneous events are not possible: $P(N((t, t+h]) \geq 2) = o(h), h \to 0$

Then $X_t := N((0, t])$ is a Poisson process with rate $\lambda$. 
Transition probabilities of the Poisson process

Let \((X_t)_{t \geq 0}\) be the Poisson process.

Define the transition probability functions

\[ P_{ij}(h) = P(X_{t+h} = j \mid X_t = i), \quad i, j \in \{0, 1, 2, \ldots \}, \quad t \geq 0, \quad h > 0 \]

What are the infinitesimal (small \(h\)) transition probability functions for \((X_t)_{t \geq 0}\)? As \(h \to 0\),

\[ P_{ii}(h) = P(X_{t+h} = i \mid X_t = i) \]

\[ = P(X_{t+h} - X_t = 0 \mid X_t = i) = P(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h) \]

\[ P_{i,i+1}(h) = P(X_{t+h} = i+1 \mid X_t = i) = P(X_{t+h} - X_t = 1) = \lambda h + o(h) \]

\[ \sum_{j \neq \{i,i+1\}} P_{ij}(h) = o(h) \]
Poisson process and transition probabilities

To sum up: \((X_t)_{t \geq 0}\) is a MC with (infinitesimal) transition probabilities satisfying

\[
P_{ii}(h) = 1 - \lambda h + o(h)
\]

\[
P_{i,i+1}(h) = \lambda h + o(h) \quad \text{as } h \to 0
\]

\[
\sum_{j \not\in \{i,i+1\}} P_{ij}(h) = o(h)
\]

What if we allow \(P_{ij}(h)\) depend on \(i\)?

\(\Rightarrow\) birth and death processes
Pure birth processes

Def. Let \((\lambda_k)_{k \geq 0}\) be a sequence of positive numbers. We define a pure birth process as a Markov process \((X_t)_{t \geq 0}\) whose stationary transition probabilities satisfy

1. \(P_{k,k+1}(h) = \lambda_k h + o(h)\) as \(h \to 0^+\)
2. \(P_{k,k}(h) = 1 - \lambda_k h + o(h)\)
3. \(P_{k,j}(h) = 0\) for \(j < k\)
4. \(X_0 = 0\)

Related model. Yule process: \(\lambda_k = \beta k\) for some \(\beta > 0\).

Describes the growth of a population - birth rate is proportional to the size of the population
Birth processes and related differential equations

Now define $P_n(t) = P(X_t = n)$. For small $h > 0$

$$P_n(t+h) = P(X_{t+h} = n) = \sum_{k=0}^{n} P(X_{t+h} = n \mid X_t = k) \cdot P(X_t = k)$$

$$= \sum_{k=0}^{n} P_{k,n}(h) \cdot P(X_t = k)$$

$$= P_{n,n}(h) \cdot P_n(t) + P_{n-1,n}(h) \cdot P_{n-1}(t) + \sum_{k=0}^{n-2} P_{k,n}(h) \cdot P(X_t = k)$$

$$= (1 - \lambda n h) P_n(t) + \lambda n h P_{n-1}(t) + o(h)$$

$$= P_n(t) - \lambda n h P_n(t) + \lambda n h P_{n-1}(t) + o(h)$$

$$P_n(t+h) - P_n(t) = -\lambda n h P_n(t) + \lambda n h P_{n-1}(t) + o(h)$$

$$P'_n(t) = \lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h} = -\lambda n P_n(t) + \lambda n h P_{n-1}(t)$$
Birth processes and related differential equations

\( P_n(t) \) satisfies the following system of differential eqns. with initial conditions

\[
\begin{align*}
\dot{P}_0(t) &= -\lambda_0 P_0(t) & P_0(0) &= 1 \\
\dot{P}_1(t) &= -\lambda_1 P_1(t) + \lambda_0 P_0(t) & P_1(0) &= 0 \\
\dot{P}_2(t) &= -\lambda_2 P_2(t) + \lambda_1 P_1(t) & P_2(0) &= 0 \\
\vdots & & \vdots \\
\dot{P}_n(t) &= -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) & P_n(0) &= 0 \\
\end{align*}
\]

Solving this system gives the p.m.f. of \( X_t \) for any \( t \)

\[ P(X_t = k) = P_k(t) \]
Description of the birth processes via sojourn times

\[(X_t)_{t \geq 0}\]

\[S_0, S_1, S_2, S_3, \ldots\]

- \(W_i\) - i-th "birth time"
- \(S_i\) - "time between (i-1)-th birth and i-th birth"

\[W_i = \sum_{k=0}^{i-1} S_k\]

Alternative way of characterizing \((X_t)_{t \geq 0}\):
- describe the distribution of \((S_i)_{i \geq 0}\)
- describe the jumps \(X_{w_{i+1}} - X_{w_i}\)
Description of the birth processes via sojourn times

Theorem

Let \((\lambda_k)_{k \geq 0}\) be a sequence of positive numbers. Let \((X_t)_{t \geq 0}\) be a non-decreasing right-continuous process, \(X_0 = 0\), taking values in \(\{0, 1, 2, \ldots\}\). Let \((S_i)_{i \geq 0}\) be the sojourn times associated with \((X_t)_{t \geq 0}\), and define \(W_t = \sum_{i=1}^{\xi - 1} S_i\).

Then conditions

(a) \(S_0, S_1, S_2, \ldots\) are independent exponential r.v.s of rate \(\lambda_0, \lambda_1, \lambda_2, \ldots\)

(b) \(X_{\omega_i} = i\) (jumps of magnitude 1)

are equivalent to

(c) \((X_t)_{t \geq 0}\) is a pure birth process with parameters \((\lambda_k)_{k \geq 0}\).
Explosion

$(X_t)_{t \geq 0}$

---

**Theorem.** Let $(X_t)_{t \geq 0}$ be a pure birth process of rates $(\lambda_k)_{k \geq 0}$. Then:

- if $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} < \infty$, then $P((X_t)_{t \geq 0} \text{ explodes}) = 1$
- if $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} = \infty$, then $P((X_t)_{t \geq 0} \text{ does not explode}) = 1$

**Hint.** $E\left( \sum_{k=0}^{\infty} S_k \right) = \sum_{k=0}^{\infty} \frac{1}{\lambda_k}$
Solving the system of differential equations (*)

\[
\begin{align*}
    P_0'(t) &= -\lambda_0 P_0(t), & P_0(0) &= 1 \\
    P_n'(t) &= -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), & P_n(0) &= 0 \quad \text{for } n \geq 1
\end{align*}
\]

\(P_0(t)\):

\[
\begin{align*}
P_0'(t) &= -\lambda_0 P_0(t) \\
\frac{P_0'(t)}{P_0(t)} &= -\lambda_0 & \left(\log \left( P_0(t) \right)\right)' &= \frac{1}{P_0(t)} - P_0'(t) = \frac{P_0'(t)}{P_0(t)} \\
g'(t) &= -\lambda_0 & g(t) &= -\lambda_0 t + K = \log \left( P_0(t) \right) \\
\Rightarrow P(t) &= e^K e^{-\lambda_0 t} = C e^{-\lambda_0 t}, & C > 0 & \Rightarrow P_0(t) &= e^{-\lambda_0 t} \\
P_0(0) &= C = 1 & \Rightarrow C = 1
\end{align*}
\]
Solving the system of differential equations (*)

$P_n(t), \ n \geq 1$

Consider the function $Q_n(t) = e^{\lambda_n t} P_n(t)$

$$(Q_n(t))' = (e^{\lambda_n t} P_n(t))' = \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} (P_n(t))'$$

$$= \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} (-\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t))$$

$$= \lambda_{n-1} e^{\lambda_n t} P_{n-1}(t)$$

$$Q_n(t) = \int_0^t \lambda_{n-1} e^{\lambda_{n-1} s} P_{n-1}(s) \, ds$$

$L_1 \ P_n(t) = e^{-\lambda_n t} \int_0^t \lambda_{n-1} e^{\lambda_{n-1} s} P_{n-1}(s) \, ds$ ← apply recursively

$P_1(t) = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{\lambda_1 s} e^{-\lambda_0 s} \, ds = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{(\lambda_1-\lambda_0)s} \, ds$ (if $\lambda_1 \neq \lambda_0$)

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0} (e^{(\lambda_1-\lambda_0)t} - 1) = \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{\lambda_1 t}$$