Today: Renewal processes.
Examples
> Q&A: May 4

Next: PK 7.4-7.5

This week:

- Homework 4 (due Friday, May 8, 11:59 PM)
- Regrades Midterm 1 (Tuesday, May 5, 8AM-11PM)
- Regrades HW2 (Tuesday, May 5, 8AM-11PM)
Poisson process as a renewal process

The Poisson process $N(t)$ with rate $\lambda > 0$ is a renewal process with $F(x) = 1 - e^{-\lambda x}$.

- Sojourn times $S_i$ are i.i.d., $S_i \sim \text{Exp}(\lambda)$.
- $S_i$ represent intervals between two consecutive events (arrivals of customers).
- $W_n = \sum_{i=0}^{n-1} S_i$.
- We can take $X_i = S_{i-1}$ in the definition of the renewal process.
Poisson process as a renewal process

We know that $N(t) \sim \text{Pois}(\lambda t)$, so in particular

$$E(N(t)) = \lambda t$$

**Example** Compute $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ for PP

$$F_2(t) = \int_0^t (1 - e^{-\lambda(t-x)}) \lambda e^{-\lambda x} \, dx = 1 - e^{-\lambda t} - \lambda \int_0^t e^{-\lambda t} \, dt = F(t) - \lambda t e^{-\lambda t}$$

Denote $\psi_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}:

$$\psi_k \ast F(t) = \int_0^t \frac{\lambda^k (t-x)^k}{k!} e^{-\lambda (t-x)} \lambda e^{-\lambda x} \, dx = \psi_{k+1}(t)$$

$F \ast F(t) = F(t) - \psi_1(t)$

$F \ast 2(t) = (F - \psi_1) \ast F(t) = F(t) - \psi_1(t) - \psi_2(t)$

$$\vdots$$

$F^{*n}(t) = F(t) - \psi_1(t) - \cdots - \psi_{n-1}(t)$
Poisson process as a renewal process (cont.)

\[
\sum_{n=1}^{\infty} F^{*n}(t) = \sum_{n=1}^{\infty} \left[ 1 - \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \right] = e^{-\lambda t} \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!}
\]

\[
= e^{-\lambda t} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{k!} = e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!}
\]

\[
= \lambda t
\]

\[M(t) = \lambda t\]
Renewal density

Proposition: Let $N(t)$ be a renewal process with continuous interrenewal times $X_i$ having density $f(x)$. Denote

$$m(t) = \sum_{n=1}^{\infty} f^*(n)(t).$$

Then

$$M(t) = \int_{0}^{t} m(x) \, dx$$

and

$$m(t) = f(t) + m^* f(t) \quad (\star)$$

renewal density

Proof: \[\frac{d}{dt} F^*(n)(t) = \left(\frac{d}{dt} F^{*(n-1)}(t)\right) \ast f(t) = f^*(n)(t) \]

Example: Compute the renewal density for PP using $(\star)$.\n
$f(x) = \lambda e^{-\lambda x}$, so $(\star)$ becomes

$$m(t) = \lambda e^{-\lambda t} + \int_{0}^{t} m(t-x) \lambda e^{-\lambda x} \, dx = \lambda e^{-\lambda t} + \int_{0}^{t} m(x) \lambda e^{-\lambda(t-x)} \, dx$$

$$= \lambda e^{-\lambda t} \left( 1 + \int_{0}^{t} m(x) e^{\lambda x} \, dx \right)$$
\[
\begin{align*}
\text{(cont.)} & \\
e^{\lambda t} m(t) &= \lambda \left( 1 + \int_0^t e^{\lambda x} m(x) \, dx \right) & \leftarrow \text{differentiate} \\
\frac{d}{dt} \left( e^{\lambda t} m(t) \right) &= \lambda \left( e^{\lambda t} m(t) \right) \\
\text{with } m(0) = \lambda \\
\Rightarrow e^{\lambda t} m(t) &= \lambda e^{\lambda t} \\

\text{Indeed,} & \\
M(t) &= \int_0^t \lambda \, dt
\end{align*}
\]
Excess life and current life of PP (summary)

Recall: Let $N(t)$ be a renewal process.

**Def.** We call

- $\gamma_t := W_{N(t)} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

**Remarks**
1) $\gamma_t > 0$ iff $N(t+h) = N(t)$
2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$
Excess life and current life of PP

Let \( N(t) \) be a PP. Then

- **excess life** \( \gamma \sim \text{Exp(}\lambda) \)
  \[
P(\gamma_t > x) = P(N(t+x) - N(t) = 0) = P(N(x) = 0) = e^{-\lambda x}
  \]

- **current life** \( \delta_t \)
  \[
P(\delta_t > x) = \begin{cases} 
  0, & \text{if } x \geq t \\
  P(N(t) - N(t-x) = 0) = e^{-\lambda x}, & x < t
  \end{cases}
  \]

- **total life** \( \beta_t = \gamma_t + \delta_t \)
  \[
  E(\gamma_t + \delta_t) = \frac{1}{\lambda} + E(\delta_t) = \frac{1}{\lambda} + \int_0^\infty P(\delta_t > x) \, dx
  \]
  \[
  = \frac{1}{\lambda} + \int_0^t e^{-\lambda x} \, dx = \frac{1}{\lambda} + \frac{1}{\lambda} (1 - e^{-\lambda t}) \rightarrow \frac{2}{\lambda}, \quad t \to \infty
  \]
Excess life and current life of PP (cont.)

- Joint distribution of \((y_t, \delta_t)\)

\[
P(y_t > x, \delta_t > y) = \begin{cases} 
0 & \text{if } y > t \\
\Pr(N(t+x) - N(t-y) = 0) = e^{-\lambda(x+y)}, & y \leq t
\end{cases}
\]

\[= y_t \text{ and } \delta_t \text{ are independent (for PP)}\]
Other renewal processes

- Traffic flow: distances between successive cars are assumed to be i.i.d. random variables.

- Counter process: particles/signals arrive on a device and lock it for time \( \tau \); particles arrive according to a PP; times at which the counter unlocks form a renewal process.
Other renewal processes

- more generally, if a component has two states (0/1, operating/non-operating etc.), switches between them, times spent in 0 are \( X_i \), times spent in 1 are \( Y_i \), \((X_i)_{i=1}^\infty, \text{i.i.d.}, (Y_i)_{i=1}^\infty, \text{i.i.d.}\), then the times of switching from 0 to 1 form a renewal process with inter-renewal times \( X_i + Y_i \)
Other renewal processes

- Markov chains: if \((Y_n)n_{n=0}, Y_n \in \{0,1,\ldots\}\) is a recurrent MC starting from \(Y_0 = k\), then the times of returns to state \(k\) form a renewal process. More precisely define

\[
W_1 = \min \{ n > 0 : Y_n = k \}
\]

\[
W_p = \min \{ n > W_{p-1} : Y_n = k \}
\]

Example with \(k=2\)

Similarly for continuous time MCs.

Strong Markov property!
Other renewal processes

- Queues. Consider a single-server queueing process

  customers arriving

  server busy/idle
  service time

(i) if customer arrival times form a renewal process
then the times of the starts of successive idle periods
  generate a second renewal time
(ii) if customers arrive according to a Poisson process,
then the times when the server passes from
  busy to free form a renewal process
Asymptotic behavior
Asymptotic behavior of renewal processes

Let $N(t)$ be a renewal process with interrenewal times $X_i, \ X_i \in (0, \infty)$.

**Thm.**

\[
P(\lim_{t \to \infty} N(t) = \infty) = 1
\]

**Proof.** $N(t)$ is nondecreasing, therefore $\exists \lim_{t \to \infty} N(t) =: N_\infty$

$N_\infty$ is the total number of events ever happened.

$N_\infty \leq k$ if and only if $W_{k+1} = \infty$

if and only if $X_i = \infty$ for some $i \leq k+1$

\[
P(N_\infty < \infty) = P(X_i = \infty \text{ for some } i) \leq \sum_{i=1}^\infty P(X_i = \infty) = 0
\]

**Thm** (Pointwise renewal thm).

\[
P(\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu}) = 1 \quad (\mu = E(X_1))
\]
Elementary Renewal Theorem

**Thm.** If \( M(t) = E(N(t)) \) and \( E(X_1) = \mu \), then
\[
\lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\mu}
\]

**Proof.** (Only for bounded \( X_i \): \( \exists K \) s.t. \( P(X_i \leq K) = 1 \))

First note that \( W_{N(t)+1} = t + Y_t \)

In lecture 8 we showed that
\[
E(W_{N(t)+1}) = \mu(M(t) + 1)
\]

so
\[
M(t) = \frac{t + E(Y_t)}{\mu} - 1
\]

\[
\frac{M(t)}{t} = \frac{1}{\mu} + \frac{1}{t} \left( \frac{E(Y_t)}{\mu} - 1 \right) \to \frac{1}{\mu} \text{ as } t \to \infty
\]

If \( X_i \leq K \), then \( Y_t \leq K =) E(Y_t) \leq K \)

**Exercise:** \((X_n)_{n \geq 0} \): 1) \( P(\lim_{n \to \infty} X_n = 0) = 1 \) 2) \( \lim_{n \to \infty} E(X_n) \geq c > 0 \)
Asymptotic distribution of $N(t)$

**Thm.** Let $N(t)$ be a renewal process with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$, then

1) \[ \lim_{t \to \infty} \frac{\text{Var}(N(t))}{t} = \frac{\sigma^2}{\mu^3} \]

2) \[ \lim_{t \to \infty} P \left( \frac{N(t) - E(N(t))}{\sqrt{\text{Var}(N(t))}} \leq x \right) \]

\[ = \lim_{t \to \infty} P \left( \frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{\sigma^2}{\mu^3} t}} \leq x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy \]

No proof. \[ N(t) \approx \frac{t}{\mu} + \sqrt{\frac{\sigma^2}{\mu^3} t} \, Z, \quad \text{where} \quad Z \sim N(0,1) \quad \text{for large} \ t \]