Problem 1  Matt has a biased coin that is more likely to come up heads than tails. He flips this coin \( n \) times and counts the number of tails. Show that this number is more likely to be even than it is to be odd.

Solution: Let \( p > 1/2 \) be the probability that a single flip comes up heads. Let \( a_n \) be the probability that we have an even number of tails after \( n \) flips. Clearly \( a_0 = 1 \). For \( n > 1 \), we have an even number of tails if and only if we either had an even number after \( n - 1 \) flips and got a heads on the \( n^{th} \) flip, or had an odd number after \( n - 1 \) flips and got a tails. Thus,

\[
a_n = pa_{n-1} + (1 - p)(1 - a_{n-1}) = (1 - p) + (2p - 1)a_{n-1}.
\]

From here it is easy to see by induction on \( n \) that \( a_n = 1/2 + (2p - 1)^n/2 > 1/2 \).
Problem 2 Let $n \geq 2$ be a positive integer and $X$ be a set containing $n^2$ consecutive numbers. Let $A$ be a subset of $X$ with $n$ elements. Show that $X \setminus A$ contains at least one arithmetic progression with $n$ elements.

Solution: Arrange the numbers in an $n \times n$ table in increasing order in each row and each column. If $A$ "misses" a row or a column, we have an arithmetic progression; else, if $A$ has exactly one element in each row/column, index with $i_1$ the column where the element of $A$ is in row 1, with $i_2$ the column where the element of $A$ is in row 2, etc. If any $i_{k+1} \geq i_k$, we have an arithmetic progression between the rows $k$ and $k + 1$. Else the only way to do it is by taking the anti-diagonal; in which case, the elements in the first upper-anti-diagonal (positions $(1, (n - 1)), (2, (n - 2)), \ldots, ((n - 1), 1))$ and $((n - 1), n)$ give an arithmetic progression.
**Problem 3** Let \( n > 0 \) be an integer. It is known that the difference \( d \) of two divisors of \( 55^n \) is a power of 2. Show that \( d = 4 \).

**Solution:** Let \( 1 \leq d_1 < d_2 \) be the two divisors whose difference is \( d \). Write

\[
d = d_2 - d_1 = 2^c.
\]

The divisors \( d_1 \) and \( d_2 \) of \( 55^{50} \) can only contain the primes 5 and 11 in their factorization. Furthermore, \( d_1 \) and \( d_2 \) cannot be both divisible by 5 at the same time and they cannot be both divisible by 11 at the same time since their difference \( d = 2^c \) is divisible neither by 5 nor by 11. Therefore,

\[
d_1 = 5^a, d_2 = 11^b \text{ or } d_1 = 11^a, d_2 = 5^b.
\]

We analyze the equations

\[
5^a - 11^b = 2^c \text{ and } 11^a - 5^b = 2^c.
\]

We show \( c = 2 \).

(i) Assume \( 11^a - 5^b = 2^c \) holds.

- Reducing the equation mod 5 we obtain

\[
2^c \equiv 1 \mod 5.
\]

Inspecting \( c \mod 4 \), we obtain \( c \equiv 0 \mod 4 \), so in particular

\[
c \text{ is even.}
\]

- Noting that \( 11^a - 5^b \) is even we obtain that \( c > 0 \) so \( c \geq 2 \).
- Reducing the equation mod 4 we obtain

\[
(-1)^a - 1 \equiv 0 \mod 4 \implies a \text{ even.}
\]

- Reducing the equation mod 3 we obtain

\[
(-1)^a - (-1)^b \equiv (-1)^c \mod 3.
\]

This cannot hold since \( a \) is even and \( c \) is even.

(ii) Assume \( 5^a - 11^b = 2^c \) holds.

- Reducing mod 3 we find

\[
(-1)^a - (-1)^b \equiv (-1)^c \mod 3
\]

which shows \( a, b \) cannot have the same parity.
- We reduce mod 8 to show that $c = 2$ is the only possibility.

If $a$ is even and $b$ is odd, write

$$ a = 2k; \quad b = 2\ell + 1, $$

and note that

$$ 5^a - 11^b = 25^k - 11 \cdot 121^\ell \equiv 1 - 11 \cdot 1 \mod 8 \equiv 6 \mod 8 \implies 2^c \equiv 6 \mod 8. $$

By inspection, this is impossible for $c \leq 2$. For $c \geq 3$ we obtain a contradiction since $2^c \equiv 0 \mod 8$.

If $a$ is odd and $b$ is even, write

$$ a = 2k + 1, \quad b = 2\ell, $$

and note that

$$ 5^a - 11^b = 5 \cdot 25^k - 121^\ell \equiv 5 \cdot 1 - 1 \equiv 4 \mod 8 \implies 2^c \equiv 4 \mod 8. $$

For $c \geq 3$ we obtain a contradiction since $2^c \equiv 0 \mod 8$. Thus $c \leq 2$. By inspection, $c = 2$ is the only possibility.

We showed $c = 2$ and therefore

$$ d = 2^c = 4. $$

The difference $d = 4$ can be achieved for instance for the pairs $(1, 5)$ or $(121, 125)$. 
Problem 4 Let $C$ be a set of $n$ points on a circle in the plane. Prove that amongst any set of $n + 1$ line segments between points in $C$, there exist two geometrically disjoint line segments, but that $n$ segments are not sufficient.

Solution: We prove the latter part first. In particular, we show that there is some set of $n$ lines so that any pair intersect. We can do this for example by picking one of our points $p$ and drawing the $n - 1$ segments from $p$ to each other point and the one segment between $p$’s nearest neighbors on either side. The segments through $p$ all intersect at $p$, and it is easy to see that the remaining segment intersects each of the others.

To show the other direction, for each of the $n$ points $p$, select the segment $(p, q)$ from our list where $q$ is the closest point counting clockwise from $p$ for which there is a segment. Since there are $n + 1$ segments and only $n$ points, some segment $(p, q)$ must not have been selected. Since this was not the segment selected from $p$, there must be another segment $q'$ so that we have a segment $(p, q')$ in our set and $p, q', q$ appear in that order going clockwise. Similarly, there must be a segment $(p', q)$ so that $q, p', p$ appear in that order going clockwise. Therefore, $(p, q')$ and $(p', q)$ are two segments in our set that do not intersect.